

# ORTHOGONAL MULTIPLE FLAG VARIETIES OF FINITE TYPE I : ODD DEGREE CASE

TOSHIHIKO MATSUKI

ABSTRACT. Let  $G$  be the split orthogonal group of degree  $2n+1$  over an arbitrary field  $\mathbb{F}$  of char  $\mathbb{F} \neq 2$ . In this paper, we classify multiple flag varieties  $G/P_1 \times \cdots \times G/P_k$  of finite type. Here a multiple flag variety is called of finite type if it has a finite number of  $G$ -orbits with respect to the diagonal action of  $G$  when  $|\mathbb{F}| = \infty$ .

## 1. INTRODUCTION

In [MWZ99], Magyar, Weyman and Zelevinsky classified multiple flag varieties for  $\mathrm{GL}_n(\mathbb{F})$  of finite type. In their subsequent paper [MWZ00], they classified multiple flag varieties for  $\mathrm{Sp}_{2n}(\mathbb{F})$  of finite type. (They assume  $\mathbb{F}$  is algebraically closed.)

Recently the author gave explicit orbit decomposition for an example of orthogonal case in [M13]. In this paper, we will classify multiple flag varieties of finite type for the split orthogonal group of degree  $2n+1$ .

Let  $\mathbb{F}$  be an arbitrary commutative field of char  $\mathbb{F} \neq 2$ . Let  $(\ , \ )$  denote the symmetric bilinear form on  $\mathbb{F}^{2n+1}$  defined by

$$(e_i, e_j) = \delta_{i, 2n+2-j}$$

for  $i, j = 1, \dots, 2n+1$  where  $e_1, \dots, e_{2n+1}$  is the canonical basis of  $\mathbb{F}^{2n+1}$ . Define the split orthogonal group

$$G = \{g \in \mathrm{GL}_{2n+1}(\mathbb{F}) \mid (gu, gv) = (u, v) \text{ for all } u, v \in \mathbb{F}^{2n+1}\}$$

with respect to this form. Let us write  $G = \mathrm{O}_{2n+1}(\mathbb{F})$  in this paper.

A subspace  $V$  of  $\mathbb{F}^{2n+1}$  is called isotropic if  $(V, V) = \{0\}$ . For a sequence  $\mathbf{a} = (\alpha_1, \dots, \alpha_p)$  of positive integers such that  $\alpha_1 + \cdots + \alpha_p \leq n$ , there corresponds the flag variety

$$M_{\mathbf{a}} = \{V_1 \subset \cdots \subset V_p \mid \dim V_i = \alpha_1 + \cdots + \alpha_i \text{ for } i = 1, \dots, p, (V_p, V_p) = \{0\}\}.$$

For the canonical flag

$$\mathcal{F}_0 : \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{\alpha_1} \subset \cdots \subset \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{\alpha_1 + \cdots + \alpha_p}$$

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Supported by JSPS Grant-in-Aid for Scientific Research (C) # 25400030.

in  $M_{\mathbf{a}}$ , the isotropic subgroup for  $\mathcal{F}_0$  in  $G$  is a standard parabolic subgroup  $P_{\mathbf{a}}$  consisting of elements in  $G$  of the form

$$\begin{pmatrix} A_1 & & & & & & * \\ & \ddots & & & & & \\ & & A_p & & & & \\ & & & A_0 & & & \\ & & & & A_p^* & & \\ & & & & & \ddots & \\ 0 & & & & & & A_1^* \end{pmatrix}$$

with  $A_i \in \mathrm{GL}_{\alpha_i}(\mathbb{F})$  for  $i = 1, \dots, p$  and  $A_0 \in \mathrm{O}_{2\alpha_0+1}(\mathbb{F})$  ( $\alpha_0 = n - (\alpha_1 + \dots + \alpha_p)$ ) where  $A_i^* = J_{\alpha_i}^t A_i^{-1} J_{\alpha_i}$  for  $i = 1, \dots, p$  and  $J_m$  is the  $m \times m$  matrix given by

$$J_m = \begin{pmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{pmatrix}.$$

Since  $M_{\mathbf{a}}$  is  $G$ -homogeneous, we can identify  $M_{\mathbf{a}}$  with  $G/P_{\mathbf{a}}$ .

*Remark 1.1.* Define the split special orthogonal group

$$G_0 = \{g \in G \mid \det g = 1\} (= \mathrm{SO}_{2n+1}(\mathbb{F})).$$

Then  $G = G_0 \sqcup (-I_{2n+1})G_0$ . Since  $-I_{2n+1}$  acts trivially on  $M_{\mathbf{a}}$ ,  $G_0$ -orbits on  $M_{\mathbf{a}}$  are the same as  $G$ -orbits.

Consider a multiple flag variety

$$\mathcal{M} = M_{\mathbf{a}_1} \times \dots \times M_{\mathbf{a}_k} \cong (G/P_{\mathbf{a}_1}) \times \dots \times (G/P_{\mathbf{a}_k})$$

with the diagonal  $G$ -action

$$g(\mathcal{F}_1, \dots, \mathcal{F}_k) = (g\mathcal{F}_1, \dots, g\mathcal{F}_k)$$

for  $g \in G$  and  $(\mathcal{F}_1, \dots, \mathcal{F}_k) \in M_{\mathbf{a}_1} \times \dots \times M_{\mathbf{a}_k}$ . The multiple flag variety  $\mathcal{M}$  is called of finite type if it has a finite number of  $G$ -orbits with respect to the diagonal  $G$ -action when  $|\mathbb{F}| = \infty$ .

If  $k = 2$ , then we have  $G \setminus ((G/P_{\mathbf{a}_1}) \times (G/P_{\mathbf{a}_2})) \cong P_{\mathbf{a}_2} \setminus G/P_{\mathbf{a}_1}$  by the map

$$(g_1, g_2) \mapsto g_2^{-1}g_1.$$

So it is of finite type by the Bruhat decomposition. In this paper we first show the following.

**Proposition 1.2.** *If  $k \geq 4$ , then  $\mathcal{M}$  is of infinite type.*

So we have only to consider triple flag varieties

$$\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}} = M_{\mathbf{a}} \times M_{\mathbf{b}} \times M_{\mathbf{c}}$$

with  $\mathbf{a} = (\alpha_1, \dots, \alpha_p)$ ,  $\mathbf{b} = (\beta_1, \dots, \beta_q)$  and  $\mathbf{c} = (\gamma_1, \dots, \gamma_r)$ . We may assume

$$p \leq q \leq r.$$

**Proposition 1.3.** *If  $\mathcal{T}$  is of finite type, then  $p = q = 1$ .*

So we may assume  $p = q = 1$  in the following. We may assume

$$\alpha_1 \leq \beta_1.$$

When  $r = 1$ , we may moreover assume

$$\alpha_1 \leq \beta_1 \leq \gamma_1.$$

We can classify triple flag varieties for  $G$  of finite type as follows.

**Theorem 1.4.** *Suppose  $1 = p = q \leq r$  and  $\alpha_1 \leq \beta_1$ . When  $r = 1$ , suppose  $\alpha_1 \leq \beta_1 \leq \gamma_1$ . Then  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of finite type if and only if  $(\mathbf{a}, \mathbf{b}, \mathbf{c})$  satisfies one of the following four conditions.*

- (I)  $\alpha_1 = \beta_1 = n$ .
- (II)  $\alpha_1 = 1$ .
- (III)  $r = 1$  and  $\gamma_1 = n$ .
- (IV)  $r = 2$  and  $\beta_1 = n$ .

*Remark 1.5.* (i) Let  $G$  be a simple algebraic group with parabolic subgroups  $P_1, P_2$  and  $P_3$ . Then  $G$ -orbit decomposition of the triple flag variety  $(G/P_1) \times (G/P_2) \times (G/P_3)$  is identified with  $P_3$ -orbit decomposition of the double flag variety  $(G/P_1) \times (G/P_2)$  by the map  $(g_1, g_2, g_3) \mapsto (g_3^{-1}g_1, g_3^{-1}g_2)$ .

When  $P_3$  is a Borel subgroup  $B$ , Littelmann and Stembridge classified double flag varieties with open  $B$ -orbits in [L94] and [S03]. (In [L94],  $P_1$  and  $P_2$  are maximal parabolic subgroups.) The two cases (I) and (II) in Theorem 1.4 are written in Table 1 of [L94]. So the double flag variety has an open orbit for these cases.

(ii) Suppose  $\mathbb{F}$  is an algebraically closed field of characteristic zero. Then a double flag variety  $(G/P_1) \times (G/P_2)$  has an open  $B$ -orbit if and only if it has a finite number of  $B$ -orbits by Brion-Vinberg's theorem ([B86], [V86]).

(iii) In [M13], we explicitly described  $G$ -orbit decomposition of the triple flag variety  $\mathcal{T} = \mathcal{T}_{(n), (n), (1^n)}$  (the case (I) in Theorem 1.4 with  $P_{\mathbf{c}} = B$ ) over an arbitrary field  $\mathbb{F}$  of char  $\mathbb{F} \neq 2$ .

(iv) The dimension of the flag variety  $M_{\mathbf{a}} = M_{(\alpha_1, \dots, \alpha_p)}$  is given by

$$\dim M_{\mathbf{a}} = n^2 - \frac{\alpha_1(\alpha_1 - 1)}{2} - \dots - \frac{\alpha_p(\alpha_p - 1)}{2} - (n - \alpha_1 - \dots - \alpha_p)^2.$$

If  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of finite type, then

$$\dim \mathcal{T} = \dim M_{\mathbf{a}} + \dim M_{\mathbf{b}} + \dim M_{\mathbf{c}} \leq \dim G = n(2n + 1).$$

For each case in Theorem 1.4, we can also get this inequality by direct computation. The equality  $\dim \mathcal{T} = \dim G$  holds for the following  $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ :

- (I)  $(n)(n)(1^n)$ .
- (II)  $(1)(2)(11)$ .
- (III)  $(1)(1)(1)$  ( $n = 1$ ).
- (IV)  $(1)(2)(11), (2)(2)(11), (2)(3)(11), (2)(3)(12), (2)(3)(21), (3)(4)(22), (3)(4)(12), (3)(4)(21), (3)(5)(22), (4)(5)(22), (4)(6)(23), (4)(6)(32), (5)(7)(33)$ .

This paper is organized as follows.

In Section 2, we first prove Proposition 1.2 by a similar argument as in [MWZ99] and [MWZ00]. So we have only to consider triple flag varieties in the rest of this

paper. We prove in this section that one of the four conditions in Theorem 1.4 holds if the triple flag variety is of finite type. The arguments are essentially the same as those in [MWZ99] and [MWZ00].

In Section 3, we describe representatives of  $G$ -orbits on the triple flag variety  $\mathcal{T}_{(\alpha),(\beta),(n)}$  (Theorem 3.15). This in particular implies that the triple flag varieties in (III) of Theorem 1.4 are of finite type. We fix  $\alpha$  and  $\beta$ -dimensional isotropic subspaces  $U_+$  and  $U_-$ , respectively, of  $\mathbb{F}^{2n+1}$ . Then we have only to classify  $R$ -orbits of maximally isotropic subspaces in  $\mathbb{F}^{2n+1}$  where  $R = \{g \in G \mid gU_+ = U_+ \text{ and } gU_- = U_-\}$ .

In Section 4, we prove that every triple flag variety  $\mathcal{T}_{(\alpha),(n),(\gamma_1,\gamma_2)}$  in (IV) of Theorem 1.4 is of finite type. Changing the order of flag varieties, we consider triple flag varieties of the form  $\mathcal{T}_{(\alpha_1,\alpha_2),(\beta),(n)}$ . We solve this problem in the following way. Put  $\alpha = \alpha_1 + \alpha_2$ . Then we may fix  $\alpha, \beta$  and  $n$ -dimensional isotropic subspaces  $U_+, U_-$  and  $V$ , respectively, by Theorem 3.15. Write  $R_V = \{g \in G \mid gU_+ = U_+, gU_- = U_-, gV = V\}$ . Then we have only to consider  $R_V$ -orbit decomposition of the Grassmann variety consisting of  $\alpha_1$ -dimensional subspaces of  $U_+$ . Our arguments are so complicated that we only show “finiteness” of the number of orbits for this case.

In Section 5, we prove that the triple flag variety  $\mathcal{T}_{(1),(\beta),(1^n)}$  is of finite type. This implies that every triple flag variety in (II) of Theorem 1.4 is of finite type. Changing the order of flag varieties, we consider triple flag varieties of the form  $\mathcal{T}_{(\alpha),(1),(1^n)}$ .

## 2. EXCLUSION OF MULTIPLE FLAG VARIETIES OF INFINITE TYPE

**2.1. A technical lemma.** In this section, we use the following technical lemma.

**Lemma 2.1.** *Let  $W$  be a nondegenerate subspace of  $\mathbb{F}^{2n+1}$ . Let  $W_1, \dots, W_k, W'_1, \dots, W'_k$  be subspaces of  $W$  such that*

$$W = W_1 + \dots + W_k = W'_1 + \dots + W'_k.$$

*Let  $U_1$  be an isotropic subspace of  $W^\perp$  and  $U_2, \dots, U_k$  be subspaces of  $U_1$ . Suppose*

$$g(W_i \oplus U_i) = W'_i \oplus U_i \quad \text{for } i = 1, \dots, k$$

*for a  $g \in G$ . Then we have:*

(i)  $g(W \oplus U_1) = W \oplus U_1$ .

(ii)  $gU_1 = U_1$ .

( By (i) and (ii),  $g$  induces an isometry on the factor space  $(W \oplus U_1)/U_1 \cong W$ .)

*Proof.* (i) The assertion is clear since

$$(W_1 \oplus U_1) + \dots + (W_k \oplus U_k) = (W'_1 \oplus U_1) + \dots + (W'_k \oplus U_k) = W \oplus U_1.$$

The assertion (ii) follows from (i) since the orthogonal space of  $W \oplus U_1$  is  $U_1$ .  $\square$

**2.2. The case of  $n = 1$ .** Suppose  $n = 1$ . Define one-dimensional isotropic subspaces

$$W_\lambda = \mathbb{F} \left( \lambda e_1 + e_2 - \frac{1}{2\lambda} e_3 \right)$$

of  $\mathbb{F}^3$  for  $\lambda \in \mathbb{F}^\times$ . Then the flag variety  $M = M_{(1)} \cong P^1(\mathbb{F})$  consists of  $\mathbb{F}e_1$ ,  $\mathbb{F}e_3$  and  $W_\lambda$  with  $\lambda \in \mathbb{F}^\times$ .

**Lemma 2.2.** *Let  $g$  be an element of  $G = O_3(\mathbb{F})$  such that  $g\mathbb{F}e_1 = \mathbb{F}e_1$ ,  $g\mathbb{F}e_3 = \mathbb{F}e_3$  and that  $gW_1 = W_1$ . Then  $g = \pm \text{id}$ .*

*Proof.* It follows from  $g\mathbb{F}e_1 = \mathbb{F}e_1$  and  $g\mathbb{F}e_3 = \mathbb{F}e_3$  that  $g$  is of the form

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \mu^{-1} \end{pmatrix}$$

with some  $\mu \in \mathbb{F}^\times$  and  $\varepsilon = \pm 1$ . Since  $gW_1 = W_1$ , we have  $\mu = \varepsilon$ .  $\square$

**Corollary 2.3.** *When  $n = 1$ , the multiple flag variety  $M \times M \times M \times M$  is of infinite type.*

*Remark 2.4.* If  $\mathbb{F}$  is algebraically closed, then the corollary also follows from

$$\dim(M \times M \times M \times M) = 4 > 3 = \dim O_3(\mathbb{F}).$$

**2.3. Proof of Proposition 1.2.** Write  $U_{[\ell]} = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_\ell$  and  $W = \mathbb{F}e_n \oplus \mathbb{F}e_{n+1} \oplus \mathbb{F}e_{n+2}$ . We may assume  $k = 4$  and  $\mathbf{a}_i = (\ell_i)$  for  $i = 1, \dots, 4$ . Define one-dimensional isotropic subspaces

$$W_\lambda = \mathbb{F} \left( \lambda e_n + e_{n+1} - \frac{1}{2\lambda} e_{n+2} \right)$$

of  $W$  for  $\lambda \in \mathbb{F}^\times$ . Take isotropic subspaces

$$\begin{aligned} V_1 &= \mathbb{F}e_n \oplus U_{[\ell_1-1]}, & V_2 &= \mathbb{F}e_{n+2} \oplus U_{[\ell_2-1]}, \\ V_3 &= W_1 \oplus U_{[\ell_3-1]} & \text{and} & \quad V_{4,\lambda} = W_\lambda \oplus U_{[\ell_4-1]} \end{aligned}$$

of  $\mathbb{F}^{2n+1}$ . Then  $m_\lambda = (V_1, V_2, V_3, V_{4,\lambda})$  are elements of  $\mathcal{M}$  for  $\lambda \in \mathbb{F}^\times$ . We have only to show that  $Gm_\lambda \neq Gm_\mu$  if  $\lambda \neq \mu$ .

Suppose  $gm_\lambda = m_\mu$  with some  $g \in G$ . Then  $gV_i = V_i$  for  $i = 1, 2, 3$  and  $gV_{4,\lambda} = V_{4,\mu}$ . Since

$$\mathbb{F}e_n + \mathbb{F}e_{n+2} + W_1 + W_\lambda = \mathbb{F}e_n + \mathbb{F}e_{n+2} + W_1 + W_\mu = W,$$

we have

$$g(W \oplus U) = W \oplus U \quad \text{and} \quad gU = U$$

where  $U = U_{[\max(\ell_1, \ell_2, \ell_3, \ell_4)-1]}$  by Lemma 2.1. Hence  $g$  induces an isometry  $\tilde{g}$  on the factor space  $\tilde{W} = (W \oplus U)/U \cong W$ . Let  $\pi$  denote the projection  $W \oplus U \rightarrow W$ . Then

$$\pi(V_1) = \mathbb{F}e_n, \quad \pi(V_2) = \mathbb{F}e_{n+2}, \quad \pi(V_3) = W_1, \quad \pi(V_\lambda) = W_\lambda \quad \text{and} \quad \pi(V_\mu) = W_\mu.$$

Since  $gV_i = V_i$  for  $i = 1, 2, 3$  and  $gV_{4,\lambda} = V_{4,\mu}$ , it follows that

$$\tilde{g}\mathbb{F}e_n = \mathbb{F}e_n, \quad \tilde{g}\mathbb{F}e_{n+2} = \mathbb{F}e_{n+2}, \quad \tilde{g}W_1 = W_1 \quad \text{and} \quad \tilde{g}W_\lambda = W_\mu.$$

Hence we have  $\tilde{g} = \pm \text{id}$  by Lemma 2.2 and therefore  $\lambda = \mu$ .  $\square$

**2.4. A lemma for  $O_5(\mathbb{F})$ .** Consider  $\mathbb{F}^5$  with the canonical basis  $f_1, \dots, f_5$  and the symmetric bilinear form  $(\ , \ )$  such that  $(f_i, f_j) = \delta_{i,6-j}$ . Take a flag  $\mathcal{F} : \mathbb{F}(f_1 + f_2) \subset \mathbb{F}f_1 \oplus \mathbb{F}f_2$  and isotropic subspaces  $U = \mathbb{F}f_4 \oplus \mathbb{F}f_5$ ,  $U' = \mathbb{F}(f_1 + f_3 - \frac{1}{2}f_5)$  of  $\mathbb{F}^5$ .

**Lemma 2.5.** *If  $g\mathcal{F} = \mathcal{F}$ ,  $gU = U$  and  $gU' = U'$  for a  $g \in G = O_5(\mathbb{F})$ , then  $g = \pm \text{id}$ .*

*Proof.* Since  $g(\mathbb{F}f_1 \oplus \mathbb{F}f_2) = \mathbb{F}f_1 \oplus \mathbb{F}f_2$  and since  $g(\mathbb{F}f_4 \oplus \mathbb{F}f_5) = \mathbb{F}f_4 \oplus \mathbb{F}f_5$ ,  $g$  is of the form

$$\begin{pmatrix} A & & 0 \\ & \varepsilon & \\ 0 & & J^t A^{-1} J \end{pmatrix}$$

with some  $A \in \text{GL}_2(\mathbb{F})$  and  $\varepsilon = \pm 1$  where  $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Hence  $gf_3 = \varepsilon f_3$ . It follows from  $gU' = U'$  that

$$gf_1 = \varepsilon f_1 \quad \text{and} \quad gf_5 = \varepsilon f_5.$$

It follows from the orthogonality of  $g$  that

$$gf_2 \in \mathbb{F}f_2 \quad \text{and} \quad gf_4 \in \mathbb{F}f_4.$$

Finally it follows from  $g(\mathbb{F}(f_1 + f_2)) = \mathbb{F}(f_1 + f_2)$  and  $(gf_2, gf_4) = (f_2, f_4) = 1$  that  $gf_2 = \varepsilon f_2$  and  $gf_4 = \varepsilon f_4$ . Hence  $g = \varepsilon \text{id}$ .  $\square$

**Corollary 2.6.** *The triple flag varieties  $\mathcal{T}_{(2),(1,1),(1,1)}$  and  $\mathcal{T}_{(1),(1,1),(1,1)}$  for  $G = O_5(\mathbb{F})$  are of infinite type.*

*Proof.* Suppose  $\mathbb{F}$  is infinite. Then there are infinitely many isotropic two-dimensional subspaces containing  $U'$ . So it follows from Lemma 2.5 that the triple flag variety  $\mathcal{T}_{(2),(1,1),(1,1)}$  is of infinite type. Similarly, there are infinitely many one-dimensional subspaces of  $U$ . So the triple flag variety  $\mathcal{T}_{(1),(1,1),(1,1)}$  is of infinite type.  $\square$

*Remark 2.7.* If  $\mathbb{F}$  is algebraically closed, then the corollary also follows from

$$\dim(M_{\mathbf{a}} \times M_{(1,1)} \times M_{(1,1)}) = 3 + 4 + 4 = 11 > 10 = \dim O_5(\mathbb{F})$$

for  $\mathbf{a} = (2)$  and  $(1)$ .

**2.5. Proof of Proposition 1.3.** Proposition 1.3 is equivalent to the following proposition:

**Proposition 2.8.** *If  $q \geq 2$ , then the triple flag variety  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type.*

*Proof.* We may assume  $p = 1$  and  $q = r = 2$ . So we can write

$$\mathbf{a} = (\alpha_1), \quad \mathbf{b} = (\beta_1, \beta_2) \quad \text{and} \quad \mathbf{c} = (\gamma_1, \gamma_2).$$

Define an isometric inclusion  $\iota : \mathbb{F}^5 \rightarrow \mathbb{F}^{2n+1}$  by

$$\iota(f_i) = e_{i+n-2}.$$

First suppose  $\alpha_1 \geq 2$ . Let  $M_{U'} \cong P^1(\mathbb{F})$  denote the variety consisting of two-dimensional isotropic subspaces of  $\mathbb{F}^5$  containing  $U' = \mathbb{F}(f_1 + f_3 - \frac{1}{2}f_5)$ . For an element  $W$  of  $M_{U'}$ , define

$$\begin{aligned} V_1 &= \mathbb{F}e_{n+2} \oplus \mathbb{F}e_{n+3} \oplus U_{[\alpha_1-2]}, & V_2 &= \mathbb{F}(e_{n-1} + e_n) \oplus U_{[\beta_1-1]}, \\ V_3 &= \mathbb{F}e_{n-1} \oplus \mathbb{F}e_n \oplus U_{[\beta_1+\beta_2-2]}, & V_4 &= \iota(U') \oplus U_{[\gamma_1-1]} \quad \text{and} \quad V_{5,W} = \iota(W) \oplus U_{[\gamma_1+\gamma_2-2]}. \end{aligned}$$

Then  $t_W = (V_1, (V_2 \subset V_3), (V_4 \subset V_{5,W}))$  is a triple flag contained in  $\mathcal{T}_{\mathbf{a},\mathbf{b},\mathbf{c}}$ . We can prove  $Gt_W \neq Gt_{W'}$  for two distinct  $W, W' \in M_{U'}$  as in the proof of Proposition 1.2 using Lemma 2.5.

Next suppose  $\alpha_1 = 1$ . Let  $M_U \cong P^1(\mathbb{F})$  denote the variety consisting of one-dimensional subspaces of  $U = \mathbb{F}f_4 \oplus \mathbb{F}f_5$ . For an element  $W$  of  $M_U$ , define

$$\begin{aligned} V_1 &= \iota(U'), & V_2 &= \mathbb{F}(e_{n-1} + e_n) \oplus U_{[\beta_1-1]}, & V_3 &= \mathbb{F}e_{n-1} \oplus \mathbb{F}e_n \oplus U_{[\beta_1+\beta_2-2]}, \\ V_{4,W} &= \iota(W) \oplus U_{[\gamma_1-1]} & \text{and} & & V_5 &= \mathbb{F}e_{n+2} \oplus \mathbb{F}e_{n+3} \oplus U_{[\gamma_1+\gamma_2-2]}. \end{aligned}$$

Then  $t_W = (V_1, (V_2 \subset V_3), (V_{4,W} \subset V_5))$  is a triple flag contained in  $\mathcal{T}_{\mathbf{a},\mathbf{b},\mathbf{c}}$ . We can prove  $Gt_W \neq Gt_{W'}$  for two distinct  $W, W' \in M_U$  as in the proof of Proposition 1.2 using Lemma 2.5.  $\square$

**2.6. First lemma for  $O_6(\mathbb{F})$ .** Consider  $\mathbb{F}^6$  with the canonical basis  $f_1, \dots, f_6$  and the symmetric bilinear form  $(\ , \ )$  such that  $(f_i, f_j) = \delta_{i,7-j}$  and let  $G$  denote the orthogonal group for this bilinear form. Define two-dimensional isotropic subspaces

$$\begin{aligned} U_1 &= \mathbb{F}f_1 \oplus \mathbb{F}f_2, & U_2 &= \mathbb{F}f_5 \oplus \mathbb{F}f_6, \\ U_{3,\lambda} &= \mathbb{F}(f_1 + f_3 + f_5) \oplus \mathbb{F}(\lambda f_2 - f_4 + (1 - \lambda)f_6) \end{aligned}$$

of  $\mathbb{F}^6$  for  $\lambda \in \mathbb{F}$ .

**Lemma 2.9.** *If  $gU_1 = U_1$ ,  $gU_2 = U_2$  and  $gU_{3,\lambda} = U_{3,\mu}$  for some  $g \in G = O_6(\mathbb{F})$  and  $\lambda, \mu \in \mathbb{F}$ . Then  $\lambda = \mu$  or  $1 - \mu$ .*

*Proof.* Suppose  $gU_1 = U_1$  and  $gU_2 = U_2$  for a  $g \in G$ . Then  $g$  is of the form

$$(2.1) \quad \begin{pmatrix} A & & 0 \\ & B(b, \det g) & \\ 0 & & J^t A^{-1} J \end{pmatrix}$$

with some  $A \in \text{GL}_2(\mathbb{F})$  and  $b \in \mathbb{F}^\times$ . Here

$$B(b, 1) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}, \quad B(b, -1) = \begin{pmatrix} 0 & b^{-1} \\ b & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

First suppose  $\det g = 1$  and  $gU_{3,\lambda} = U_{3,\mu}$ . Then we have

$$g\mathbb{F}(f_1 + f_3 + f_5) = \mathbb{F}(f_1 + f_3 + f_5)$$

and

$$g\mathbb{F}(\lambda f_2 - f_4 + (1 - \lambda)f_6) = \mathbb{F}(\mu f_2 - f_4 + (1 - \mu)f_6).$$

Hence

$$\begin{aligned} gf_1 &= bf_1, & gf_3 &= bf_3, & gf_5 &= bf_5, \\ g\lambda f_2 &= b^{-1}\mu f_2, & gf_4 &= b^{-1}f_4 & \text{and} & g(1 - \lambda)f_6 = b^{-1}(1 - \mu)f_6 \end{aligned}$$

with some  $b \in \mathbb{F}^\times$ . Thus we have

$$\lambda = (f_5, \lambda f_2) = (gf_5, g\lambda f_2) = (bf_5, b^{-1}\mu f_2) = \mu.$$

On the other hand, suppose  $\det g = -1$  and  $gU_{3,\lambda} = U_{3,\mu}$ . Then we have

$$g\mathbb{F}(f_1 + f_3 + f_5) = \mathbb{F}(\mu f_2 - f_4 + (1 - \mu)f_6)$$

and

$$g\mathbb{F}(\lambda f_2 - f_4 + (1 - \lambda)f_6) = \mathbb{F}(f_1 + f_3 + f_5).$$

Hence

$$\begin{aligned} gf_1 &= -b\mu f_2, & gf_3 &= bf_4, & gf_5 &= -b(1 - \mu)f_6, \\ g\lambda f_2 &= -b^{-1}f_1, & gf_4 &= b^{-1}f_3 & \text{and} & g(1 - \lambda)f_6 = -b^{-1}f_5 \end{aligned}$$

with some  $b \in \mathbb{F}^\times$ . Thus we have

$$\lambda = (f_5, \lambda f_2) = (gf_5, g\lambda f_2) = (-b(1 - \mu)f_6, -b^{-1}f_1) = 1 - \mu.$$

□

**Corollary 2.10.** *The triple flag variety  $\mathcal{T}_{(2),(2),(2)}$  for  $G = \mathrm{O}_6(\mathbb{F})$  is of infinite type.*

*Remark 2.11.* Suppose that  $\mathbb{F}$  is algebraically closed. Then we have

$$\dim \mathcal{T}_{(2),(2),(2)} = 5 + 5 + 5 = 15 = \dim G.$$

But  $\mathcal{T}_{(2),(2),(2)}$  has no open  $G$ -orbit.

**2.7. Second lemma for  $\mathrm{O}_6(\mathbb{F})$ .** Define isotropic subspaces

$$\begin{aligned} U_1 &= \mathbb{F}f_1 \oplus \mathbb{F}f_2, & U_2 &= \mathbb{F}f_5 \oplus \mathbb{F}f_6, & U_{4,\lambda} &= \mathbb{F}(\lambda f_1 - f_3 + (1 - \lambda)f_5) \\ \text{and } U_5 &= \mathbb{F}(f_1 - f_5) \oplus \mathbb{F}(f_1 - f_3) \oplus \mathbb{F}(f_2 + f_4 + f_6) \end{aligned}$$

of  $\mathbb{F}^6$  for  $\lambda \in \mathbb{F}$ . Then  $t_\lambda = (U_1, U_2, (U_{4,\lambda} \subset U_5))$  are triple flags in  $\mathcal{T} = \mathcal{T}_{(2),(2),(1,2)}$ .

**Lemma 2.12.** *If  $gt_\lambda = t_\mu$  for  $a \in G$  and  $\lambda, \mu \in \mathbb{F}$ , then  $\lambda = \mu$ .*

*Proof.* Since  $gU_1 = U_1$  and  $gU_2 = U_2$ ,  $g$  is of the form (2.1). Since  $(U_1 \oplus U_2) \cap U_5 = \mathbb{F}(f_1 - f_5)$ , it follows from  $gU_5 = U_5$  that

$$gf_1 = af_1 \quad \text{and} \quad gf_5 = af_5$$

with some  $a \in \mathbb{F}^\times$ . Finally it follows from  $gU_{4,\lambda} = U_{4,\mu}$  that  $\det g = 1$  and that  $\lambda = \mu$ . □

**Corollary 2.13.** *The triple flag variety  $\mathcal{T} = \mathcal{T}_{(2),(2),(1,2)}$  for  $G = \mathrm{O}_6(\mathbb{F})$  is of infinite type.*

*Remark 2.14.* Suppose that  $\mathbb{F}$  is algebraically closed. Let  $U'_4 = \mathbb{F}(f_1 + f_2 - f_3 + f_4 + f_6)$  and  $m' = (U_1, U_2, (U'_4 \subset U_5))$ . Then  $Gm'$  is the open  $G$ -orbit in the triple flag variety  $\mathcal{T} = \mathcal{T}_{(2),(2),(1,2)}$  for  $G = \mathrm{O}_6(\mathbb{F})$ .



**2.8. Some conditions on  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$ .** By Proposition 1.3, we may assume  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  with

$$\mathbf{a} = (\alpha_1), \quad \mathbf{b} = (\beta_1) \quad \text{and} \quad \mathbf{c} = (\gamma_1, \dots, \gamma_r).$$

**Proposition 2.15.** *Suppose that  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of finite type.*

- (i) *If  $r = 1$  and  $\alpha_1 \leq \beta_1 \leq \gamma_1$ , then  $\alpha_1 = 1$  or  $\gamma_1 = n$ .*
- (ii) *If  $r \geq 2$  and  $\alpha_1 \leq \beta_1$ , then  $\alpha_1 = 1$  or  $\beta_1 = n$ .*

*Proof.* (i) Suppose  $r = 1$  and  $1 < \alpha_1 \leq \beta_1 \leq \gamma_1 < n$ . Define an isometric inclusion  $\iota : \mathbb{F}^6 \rightarrow \mathbb{F}^{2n+1}$  by

$$\begin{aligned} \iota(f_1) &= e_{n-2}, & \iota(f_2) &= e_{n-1}, & \iota(f_3) &= e_n, \\ \iota(f_4) &= e_{n+2}, & \iota(f_5) &= e_{n+3} & \text{and} & \iota(f_6) = e_{n+4}. \end{aligned}$$

Define isotropic subspaces

$$\begin{aligned} V_1 &= \mathbb{F}e_{n-2} \oplus \mathbb{F}e_{n-1} \oplus U_{[\alpha_1-2]}, & V_2 &= \mathbb{F}e_{n+3} \oplus \mathbb{F}e_{n+4} \oplus U_{[\beta_1-2]} \\ \text{and } V_{3,\lambda} &= \iota(U_{3,\lambda}) \oplus U_{[\gamma_1-2]}. \end{aligned}$$

of  $\mathbb{F}^{2n+1}$  for  $\lambda \in \mathbb{F}$ . Then  $t_\lambda = (V_1, V_2, V_{3,\lambda})$  is a triple flag contained in  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$ . We can prove  $Gt_\lambda \neq Gt_\mu$  if  $\lambda \neq \mu, 1 - \mu$  as in the proof of Proposition 1.2 using Lemma 2.9. Hence  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type.

(ii) First suppose  $r \geq 3$  and  $1 < \alpha_1 \leq \beta_1 < n$ . Write  $\gamma = \gamma_1 + \gamma_2$ . Then we can consider the natural projection

$$M_{(\gamma_1, \dots, \gamma_r)} \rightarrow M_{(\gamma)}.$$

Since  $1 < \gamma < n$ , it follows from (i) that  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, (\gamma)}$  is of infinite type. Hence  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type.

So we have only to consider the case of  $r = 2$ . Suppose  $1 < \alpha_1 \leq \beta_1 < n$ . Then we can show that  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type as follows.

Suppose first  $\gamma_1 > 1$ . Then  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, (\gamma_1)}$  is of infinite type by (i). Hence  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type.

Next suppose  $\gamma_1 + \gamma_2 < n$ . Then  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, (\gamma_1 + \gamma_2)}$  is of infinite type by (i). Hence  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type.

So we have only to consider the case of  $\mathbf{c} = (1, n-1)$ . Let  $\iota : \mathbb{F}^6 \rightarrow \mathbb{F}^{2n+1}$  be as in (i) and define isotropic subspaces

$$\begin{aligned} V_1 &= \mathbb{F}e_{n-2} \oplus \mathbb{F}e_{n-1} \oplus U_{[\alpha_1-2]}, & V_2 &= \mathbb{F}e_{n+3} \oplus \mathbb{F}e_{n+4} \oplus U_{[\beta_1-2]}, \\ V_{4,\lambda} &= \iota(U_{4,\lambda}), & V_5 &= \iota(U_5) \oplus U_{[n-3]} \end{aligned}$$

of  $\mathbb{F}^{2n+1}$  for  $\lambda \in \mathbb{F}$ . Then  $(V_1, V_2, (V_{4,\lambda} \subset V_5))$  is a triple flag contained in  $\mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$ . We can prove  $Gt_\lambda \neq Gt_\mu$  if  $\lambda \neq \mu$  as in the proof of Proposition 1.2 using Lemma 2.12. Hence  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type.  $\square$

**2.9. A lemma for  $O_7(\mathbb{F})$ .** Consider  $\mathbb{F}^7$  with the canonical basis  $f_1, \dots, f_7$  and the symmetric bilinear form  $(\ , \ )$  such that  $(f_i, f_j) = \delta_{i,8-j}$ . Write  $U_+ = \mathbb{F}f_1 \oplus \mathbb{F}f_2$  and  $U_- = \mathbb{F}f_6 \oplus \mathbb{F}f_7$ . Take a maximal isotropic subspace

$$U = \mathbb{F}(f_2 + f_3) \oplus \mathbb{F}\left(f_1 + f_4 - \frac{1}{2}f_7\right) \oplus \mathbb{F}(f_5 - f_6)$$

in  $\mathbb{F}^7$ .

**Lemma 2.16.** *If  $g\mathbb{F}(f_1 + f_2) = \mathbb{F}(f_1 + f_2)$ ,  $gU_+ = U_+$ ,  $gU_- = U_-$  and  $gU = U$  for  $g \in G = \mathrm{O}_7(\mathbb{F})$ , then  $g = \pm \mathrm{id}$ .*

*Proof.* It follows from  $gU_+ = U_+$  and  $gU_- = U_-$  that  $g$  is of the form

$$\begin{pmatrix} A & & 0 \\ & B & \\ 0 & & J^t A^{-1} J \end{pmatrix}$$

with some  $A \in \mathrm{GL}_2(\mathbb{F})$  and  $B \in \mathrm{O}_3(\mathbb{F})$ . It follows from  $gU = U$  that

$$\begin{aligned} gf_2 &= af_2, & gf_3 &= af_3, & gf_5 &= a^{-1}f_5, & gf_6 &= a^{-1}f_6, \\ gf_1 &= \varepsilon f_1, & gf_4 &= \varepsilon f_4 & \text{and} & gf_7 &= \varepsilon f_7 \end{aligned}$$

with some  $a \in \mathbb{F}^\times$  and  $\varepsilon = \pm 1$ . Finally it follows from  $g\mathbb{F}(f_1 + f_2) = \mathbb{F}(f_1 + f_2)$  that  $a = \varepsilon$ . Hence  $g = \pm \mathrm{id}$ .  $\square$

Let  $M_{U_+} \cong P^1(\mathbb{F})$  denote the variety consisting of three-dimensional (maximal) isotropic subspaces in  $\mathbb{F}^7$  containing  $U_+$ .

**Corollary 2.17.** *The triple flag variety  $\mathcal{T} = \mathcal{T}_{(2),(3),(1,1,1)}$  for  $G = \mathrm{O}_7(\mathbb{F})$  is of infinite type.*

*Proof.* For an element  $W \in M_{U_+}$ , we can take a triple flag

$$t_W = (U_-, U, (\mathbb{F}(f_1 + f_2) \subset U_+ \subset W))$$

in  $\mathcal{T}$ . By lemma 2.16, we have  $Gt_W \neq Gt_{W'}$  for two distinct element  $W$  and  $W'$  in  $M_{U_+}$ .  $\square$

*Remark 2.18.* If  $\mathbb{F}$  is algebraically closed, then the corollary also follows from

$$\dim \mathcal{T} = 7 + 6 + 9 = 22 > 21 = \dim \mathrm{O}_7(\mathbb{F}).$$

**2.10. Final exclusion.** As in Proposition 2.15, we may assume that  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  with

$$\mathbf{a} = (\alpha_1), \quad \mathbf{b} = (\beta_1) \quad \text{and} \quad \mathbf{c} = (\gamma_1, \dots, \gamma_r).$$

We may moreover assume  $\alpha_1 \leq \beta_1$ .

**Proposition 2.19.** *Suppose  $r \geq 3$  and  $\mathcal{T}$  is of finite type. Then  $\alpha_1 = 1$  or  $\alpha_1 = \beta_1 = n$ .*

*Proof.* We may assume  $r = 3$ . Suppose  $\alpha_1 > 1$ . Then by Proposition 2.15 (ii), we have  $\beta_1 = n$ . Suppose  $\alpha_1 < n$ . Define an isometric inclusion  $\iota : \mathbb{F}^7 \rightarrow \mathbb{F}^{2n+1}$  by  $\iota(f_i) = e_{i+n-3}$ . Then we can define isotropic subspaces

$$\begin{aligned} V_1 &= \mathbb{F}e_{n+3} \oplus \mathbb{F}e_{n+4} \oplus U_{[\alpha_1-2]}, & V_2 &= \iota(U) \oplus U_{[\beta_1-3]}, \\ V_3 &= \mathbb{F}(e_{n-2} + e_{n-1}) \oplus U_{[\gamma_1-1]}, & V_4 &= \mathbb{F}e_{n-2} \oplus \mathbb{F}e_{n-1} \oplus U_{[\gamma_1+\gamma_2-2]} \end{aligned}$$

$$\text{and } V_{5,W} = \iota(W) \oplus U_{[\gamma_1+\gamma_2+\gamma_3-3]}$$

of  $\mathbb{F}^{2n+1}$  for  $W \in M_{U_+}$ . Then the triple flags  $t_W = (V_1, V_2, (V_3 \subset V_4 \subset V_{5,W}))$  are contained in  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  for  $W \in M_{U_+}$ . We can prove  $Gt_W \neq Gt_{W'}$  if  $W$  and  $W'$  are

two distinct elements in  $M_{U_+}$  as in the proof of Proposition 1.2 using Lemma 2.16. Hence  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of infinite type.  $\square$

**2.11. Conclusion.** Combining Proposition 2.15 and Proposition 2.19, we have the following theorem.

**Theorem 2.20.** *Suppose  $1 = p = q \leq r$  and  $\alpha_1 \leq \beta_1$ . When  $r = 1$ , suppose  $\alpha_1 \leq \beta_1 \leq \gamma_1$ . If  $\mathcal{T} = \mathcal{T}_{\mathbf{a}, \mathbf{b}, \mathbf{c}}$  is of finite type, then  $(\mathbf{a}, \mathbf{b}, \mathbf{c})$  satisfies one of the following four conditions.*

- (I)  $\alpha_1 = \beta_1 = n$ .
- (II)  $\alpha_1 = 1$ .
- (III)  $r = 1$  and  $\gamma_1 = n$ .
- (IV)  $r = 2$  and  $\beta_1 = n$ .

### 3. ORBITS ON $\mathcal{T}_{(\alpha), (\beta), (n)}$

**3.1. Preliminaries.** First we prepare some general notations and results. Write  $\bar{i} = 2n + 2 - i$  for  $i \in I = \{1, \dots, 2n + 1\}$ . For  $d \leq n$ , let

$$W = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_d, \quad \overline{W} = \mathbb{F}e_{\bar{d}} \oplus \dots \oplus \mathbb{F}e_{\bar{1}}, \quad U = \mathbb{F}e_{d+1} \oplus \dots \oplus \mathbb{F}e_{\bar{d+1}}$$

and  $m = 2n + 1 - 2d = \dim U$ . For  $A \in \mathrm{GL}_d(\mathbb{F})$  and  $B \in \mathrm{O}_m(\mathbb{F})$ , define elements

$$\ell(A) = \begin{pmatrix} A & & 0 \\ & I_m & \\ 0 & & J_d^t A^{-1} J_d \end{pmatrix} \quad \text{and} \quad \ell_0(B) = \begin{pmatrix} I_d & & 0 \\ & B & \\ 0 & & I_d \end{pmatrix}$$

of  $G$ . As in [M13], the maximal parabolic subgroup  $P_W = \{g \in G \mid gW = W\}$  is decomposed as  $P_W = L_W L_U N_W = N_W L_W L_U$  where

$$L_W = \{\ell(A) \mid A \in \mathrm{GL}_d(\mathbb{F})\}, \quad L_U = \{\ell_0(B) \mid B \in \mathrm{O}_m(\mathbb{F})\}$$

$$\text{and} \quad N_W = \left\{ \begin{pmatrix} I_d & * & * \\ 0 & I_m & * \\ 0 & 0 & I_d \end{pmatrix} \in G \right\}.$$

The subgroup  $L_W L_U \cong L_W \times L_U$  is a Levi subgroup of  $P_W$  and  $N_W$  is the unipotent radical of  $P_W$  consisting of elements of the form

$$(3.1) \quad g(X, Z) = \begin{pmatrix} I_d & X & Z \\ 0 & I_m & -J_m^t X J_d \\ 0 & 0 & I_d \end{pmatrix}$$

with matrices  $X = \{x_{i,j}\}_{j=1, \dots, m}^{i=1, \dots, d}$  and  $Z = \{z_{i,j}\}_{j=1, \dots, d}^{i=1, \dots, d}$  satisfying

$$(3.2) \quad Z + J_d^t Z J_d = -X J_m^t X J_d.$$

So we have a bijection between  $\mathbb{F}^{dm+d(d-1)/2}$  and  $N_W$  given by

$$(X, \{z_{i,j}\}_{i+j \leq d}) \mapsto g(X, Z).$$

Put  $Y = -J_m^t X J_d$ . Then  $g(X, Z)$  is rewritten as

$$(3.3) \quad g(X, Z) = g'(Y, Z) = \begin{pmatrix} I_d & -J_d^t Y J_m & Z \\ 0 & I_m & Y \\ 0 & 0 & I_d \end{pmatrix}.$$

The conditon (3.2) is rewritten as

$$(3.4) \quad Z + J_d^t Z J_d = -J_d^t Y J_m Y.$$

A pair of isotropic subspaces  $(V, V')$  is called nondegenerate if

$$V \cap V'^\perp = V' \cap V^\perp = \{0\}.$$

Let  $W'$  be an isotropic subspace of  $\mathbb{F}^{2n+1}$  such that  $(W, W')$  is nondegenerate. Then  $W'$  is written as

$$W' = \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_d$$

with vectors

$$v_j = e_{2n+1-d+j} + \sum_{i=1}^d z_{i,j} e_i + \sum_{i=1}^m y_{i,j} e_{d+i}.$$

Define matrices  $Z = \{z_{i,j}\}$  and  $Y = \{y_{i,j}\}$ . Since the condition (3.4) is equivalent to  $(v_i, v_j) = 0$  for  $i, j = 1, \dots, d$ , we have:

**Lemma 3.1.**  $g'(Y, Z) \in G = \text{O}_{2n+1}(\mathbb{F})$ .

**Corollary 3.2.** Let  $W = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_d$  with  $d \leq n$  and let  $W'$  be an isotropic subspace of  $\mathbb{F}^{2n+1}$  such that  $(W, W')$  is nondegenerate. Then there exists an  $h \in G$  such that

$$hW' = \overline{W} = \mathbb{F}e_{\overline{d}} \oplus \cdots \oplus \mathbb{F}e_{\overline{1}}$$

and that  $h$  acts trivially on  $W$ .

*Proof.*  $h = g'(Y, Z)^{-1}$  is a desired element. □

**Corollary 3.3.** Let  $(V, V')$  be a nondegenerate pair of isotropic subspaces in  $\mathbb{F}^{2n+1}$  with  $\dim V = \dim V' = \ell$ . Then there exists a  $g \in G$  such that

$$gV = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_\ell \quad \text{and that} \quad gV' = \mathbb{F}e_{\overline{\ell}} \oplus \cdots \oplus \mathbb{F}e_{\overline{1}}.$$

Let  $p_{\overline{W}}$  denote the projection

$$p_{\overline{W}} : \mathbb{F}^{2n+1} = W^\perp \oplus \overline{W} \rightarrow \overline{W}.$$

**Lemma 3.4.** Let  $V$  be a maximally isotropic subspace in  $\mathbb{F}^{2n+1}$ . Then

- (i)  $p_{\overline{W}}(V)$  is the orthogonal space of  $W \cap V$  in  $\overline{W}$ .
- (ii)  $(W^\perp \cap V)/(W \cap V)$  is maximally isotropic in  $W^\perp/W$ .

*Proof.* The spaces  $W \cap V$  and  $p_{\overline{W}}(V)$  are orthogonal since  $W \perp W^\perp$  and  $V$  is isotropic. Hence we have

$$(3.5) \quad \dim(W \cap V) + \dim p_{\overline{W}}(V) \leq \dim W$$

since  $(W, \overline{W})$  is nondegenerate. On the other hand, we have

$$(3.6) \quad \dim((W^\perp \cap V)/(W \cap V)) \leq n - \dim W$$

since  $(W^\perp \cap V)/(W \cap V)$  is isotropic in  $W^\perp/W$ . Since

$$\begin{aligned} n = \dim V &= \dim(W^\perp \cap V) + \dim p_{\overline{W}}(V) \\ &= \dim((W^\perp \cap V)/(W \cap V)) + \dim(W \cap V) + \dim p_{\overline{W}}(V), \end{aligned}$$

we have the equalities in (3.5) and (3.6).  $\square$

We can easily rewrite Lemma 3.1 in the following form.

**Lemma 3.5.** *Let  $K$  be an index subset of  $I$  such that  $K \cap \overline{K} = \emptyset$ . Let  $v_k$  be vectors in  $\mathbb{F}^{2n+1}$  of the form*

$$v_k = e_k + \sum_{i \in I-K} c_{i,k} e_i$$

*with some  $c_{i,k} \in \mathbb{F}$  for  $k \in K$  such that  $(v_k, v_\ell) = 0$  for  $k, \ell \in K$ . Then there exists a  $g \in G$  such that*

$$ge_k = \begin{cases} e_k & \text{if } k \in \overline{K}, \\ v_k & \text{if } k \in K, \\ e_k - \sum_{i \in \overline{K}} c_{\overline{k},i} e_i & \text{if } k \in I - K - \overline{K}. \end{cases}$$

**3.2. Normalization of  $U_+$  and  $U_-$ .** Let  $U_+$  and  $U_-$  be  $\alpha$  and  $\beta$ -dimensional isotropic subspaces of  $\mathbb{F}^{2n+1}$ , respectively. Define subspaces

$$W_0 = U_+ \cap U_-, \quad W_+ = U_+ \cap U_-^\perp \quad \text{and} \quad W_- = U_- \cap U_+^\perp$$

of  $\mathbb{F}^{2n+1}$ . Write  $a_0 = \dim W_0$ ,  $a_+ = \dim W_+ - a_0$  and  $a_- = \dim W_- - a_0$ . Since the bilinear form  $(\ , \ )$  is nondegenerate on the pair  $(U_+/W_+, U_-/W_-)$ , we have

$$\alpha - a_0 - a_+ = \beta - a_0 - a_-.$$

Put  $a_1 = \alpha - a_0 - a_+ = \beta - a_0 - a_-$  and  $d = a_0 + a_+ + a_-$ . Define subspaces

$$\begin{aligned} W_{(0)} &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{a_0}, & W_{(+)} &= \mathbb{F}e_{a_0+1} \oplus \cdots \oplus \mathbb{F}e_{a_0+a_+}, \\ W_{(-)} &= \mathbb{F}e_{a_0+a_++1} \oplus \cdots \oplus \mathbb{F}e_d, & U_{(+)} &= \mathbb{F}e_{d+1} \oplus \cdots \oplus \mathbb{F}e_{d+a_1} \\ \text{and } U_{(-)} &= \mathbb{F}e_{\overline{d+a_1}} \oplus \cdots \oplus \mathbb{F}e_{\overline{d+1}} \end{aligned}$$

of  $\mathbb{F}^{2n+1}$ . Put  $W = W_{(0)} \oplus W_{(+)} \oplus W_{(-)} = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_d$ . Let  $\overline{W}$  and  $U$  be as in Section 3.1.

**Proposition 3.6.** *There exists an element  $g \in G$  such that*

$$gU_+ = W_{(0)} \oplus W_{(+)} \oplus U_{(+)} \quad \text{and that} \quad gU_- = W_{(0)} \oplus W_{(-)} \oplus U_{(-)}.$$

*Proof.* Since  $W_+ + W_-$  is an isotropic subspace of  $\mathbb{F}^{2n+1}$  with  $\dim(W_+ + W_-) = a_0 + a_+ + a_- = d$ , we can take a  $g_1 \in G$  such that  $g_1(W_+ + W_-) = W$ . Since  $g_1W_+$  and  $g_1W_-$  are  $a_0 + a_+$ -dimensional and  $a_0 + a_-$ -dimensional subspaces, respectively, of  $W$ , there exists a  $g_2 = \ell(A)$  with some  $A \in \mathrm{GL}_d(\mathbb{F})$  such that

$$(3.7) \quad g_2g_1W_+ = W_{(0)} \oplus W_{(+)} \quad \text{and that} \quad g_2g_1W_- = W_{(0)} \oplus W_{(-)}$$

by Lemma 6.1 in the appendix.

Since the pair  $((g_2g_1U_+ + W)/W, (g_2g_1U_- + W)/W)$  is a nondegenerate pair of isotropic subspaces in the factor space  $W^\perp/W \cong U$ , we can take a  $g_3 \in \mathrm{SO}(U)$  such that

$$g_3g_2g_1U_+ + W = U_{(+)} \oplus W \quad \text{and that} \quad g_3g_2g_1U_- + W = U_{(-)} \oplus W$$

by Corollary 3.3. By (3.7) we can write

$$\begin{aligned} g_3g_2g_1U_+ &= W_{(0)} \oplus W_{(+)} \oplus \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_{a_1} \\ \text{and } g_3g_2g_1U_- &= W_{(0)} \oplus W_{(-)} \oplus \mathbb{F}v'_1 \oplus \cdots \oplus \mathbb{F}v'_{a_1} \end{aligned}$$

with some vectors  $v_1, \dots, v_{a_1}, v'_1, \dots, v'_{a_1}$  of the form

$$v_j = e_{d+j} + \sum_{i=1}^{a_-} y_{i,j} e_{a_0+a_++i} \quad \text{and} \quad v'_j = e_{d'+j} + \sum_{i=1}^{a_+} y'_{i,j} e_{a_0+i}$$

for  $j = 1, \dots, a_1$ . Define a  $d \times m$  matrix

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \{y'_{i,j}\} \\ \{y_{i,j}\} & 0 & 0 \end{pmatrix}$$

and  $g_4 = g(X, Z) \in N_W$  with a suitable  $Z$ . Then we have

$$g_4^{-1}v_j = e_{d+j} \quad \text{and} \quad g_4^{-1}v'_j = e_{d'+j}$$

for  $j = 1, \dots, a_1$ . Thus the element  $g = g_4^{-1}g_3g_2g_1 \in G$  satisfies the desired property.  $\square$

**3.3. Structure of  $R = P_{U_+} \cap P_{U_-}$ .** By Proposition 3.6, we may assume

$$U_+ = W_{(0)} \oplus W_{(+)} \oplus U_{(+)} \quad \text{and that} \quad U_- = W_{(0)} \oplus W_{(-)} \oplus U_{(-)}.$$

Write  $R = P_{U_+} \cap P_{U_-} = \{g \in G \mid gU_+ = U_+, gU_- = U_-\}$ .

**Proposition 3.7.** (i)  $R = (N_W \cap R)(L_U \cap R)(L_W \cap R)$ .

(ii)  $L_W \cap R$  consists of elements

$$\ell \left( \begin{pmatrix} A & Y_1 & Y_2 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \right)$$

with  $A \in \mathrm{GL}_{a_0}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{a_+}(\mathbb{F})$ ,  $C \in \mathrm{GL}_{a_-}(\mathbb{F})$ ,  $Y_1 \in \mathcal{M}(a_0, a_+; \mathbb{F})$  ( $= \{a_0 \times a_+$ -matrices with entries in  $\mathbb{F}\}$ ) and  $Y_2 \in \mathcal{M}(a_0, a_-; \mathbb{F})$ .

(iii)  $L_U \cap R$  consists of elements  $\ell_{00}(D, D')$  with  $D \in \mathrm{GL}_{a_1}(\mathbb{F})$  and  $D' \in \mathrm{O}_{m-2a_1}(\mathbb{F})$  where

$$\ell_{00}(D, D') = \ell_0 \left( \begin{pmatrix} D & & 0 \\ & D' & \\ 0 & & J_{a_1} {}^t D^{-1} J_{a_1} \end{pmatrix} \right).$$

(iv)  $N_W \cap R$  consists of elements  $g(X, Z) \in N_W$  satisfying

$$\begin{aligned} x_{i,j} &= 0 \quad \text{for } (i, j) \in \{a_0 + a_+ + 1, \dots, d\} \times \{1, \dots, a_1\} \\ &\quad \sqcup \{a_0 + 1, \dots, a_0 + a_+\} \times \{m - a_1 + 1, \dots, m\}. \end{aligned}$$

*Proof.* (ii), (iii) and (iv) are clear. So we have only to prove (i). Let  $g$  be an element of  $R$ . Since  $gW = W$ ,  $g$  is contained in  $P_W$ . So we can write  $g = g_N g_U g_W$  with  $g_W \in L_W$ ,  $g_U \in L_U$  and  $g_N \in N_W$ .

Since  $g_N g_U$  acts trivially on  $W$ , we have  $g_W(W_+) = W_+$  and  $g_W(W_-) = W_-$ . Hence  $g_W$  is of the form in (ii) and  $g_W \in R$ . This implies also  $g_N g_U = g g_W^{-1} \in R$ .

Let  $u$  be an element of  $U_{(+)}$ . Then  $g_U(u) \in U$ . We can write

$$g_N g_U(u) = g_U(u) + w$$

with some  $w \in W$ . On the other hand, it follows from  $g_N g_U \in R$  that

$$g_N g_U(u) \in U_+ \subset U_{(+)} \oplus W.$$

Hence we have  $g_U(u) \in U_{(+)}$ . Thus we have proved  $g_U(U_{(+)}) = U_{(+)}$ . We also have  $g_U(U_{(-)}) = U_{(-)}$  in the same way. So the element  $g_U$  is of the form in (iii) and therefore  $g_U \in R$ . This implies also  $g_N \in R$ .

Thus we have proved

$$R \subset (N_W \cap R)(L_U \cap R)(L_W \cap R).$$

The converse inclusion is trivial. □

**3.4. Invariants of the  $R$ -orbit of  $V \in M = M_{(n)}$ .** Let  $V$  be a maximal isotropic subspace in  $\mathbb{F}^{2n+1}$ . Then the following  $b_1, \dots, b_{11}$  are clearly invariants of the  $R$ -orbit of  $V$ .

$$\begin{aligned} b_1 &= \dim(W_0 \cap V), & b_2 &= a_0 - b_1, \\ b_3 &= \dim(W_+ \cap V) - b_1, & b_4 &= \dim(W_- \cap V) - b_1, \\ b_5 &= \dim(U_+ \cap V) - b_1 - b_3, & b_6 &= \dim(U_- \cap V) - b_1 - b_4, \\ b_7 &= \dim(W \cap V) - b_1 - b_3 - b_4, \\ b_8 &= \dim((W_+ + U_-) \cap V) - \dim(W \cap V) - b_6, \\ b_9 &= \dim((U_+ + W_-) \cap V) - \dim(W \cap V) - b_5, \\ b_{10} &= a_+ - b_3 - b_7 - b_8, & b_{11} &= a_- - b_4 - b_7 - b_9. \end{aligned}$$

We will deduce the remaining essential invariants from

$$X = X(V) = (U_+ + U_-) \cap V \quad \text{and} \quad X' = X'(V) = (U_+ + U_-) \cap V^\perp.$$

Consider the projection  $\pi : W^\perp \rightarrow \pi(W^\perp) = W^\perp/W$ . Then  $\pi(W^\perp)$  is decomposed as

$$\pi(W^\perp) = \pi(U_+) \oplus \pi(U_-) \oplus \pi(Z)$$

where  $Z = (U_+ + U_-)^\perp \subset W^\perp$ . Put  $a_2 = \frac{m-1}{2} - a_1 = n - d - a_1$ . Then

$$\dim \pi(Z) = 2a_2 + 1 \quad \text{and} \quad \dim Z = 2a_2 + 1 + d.$$

Let  $\pi_+, \pi_-$  and  $\pi_Z$  denote the projections of  $W^\perp$  onto  $\pi(U_+)$ ,  $\pi(U_-)$  and  $\pi(Z)$ , respectively.

**Lemma 3.8.**  $a_1 - \dim \pi(X') = a_2 - \dim \pi(Z \cap V)$ .

*Proof.* Since  $Z + V$  is the orthogonal space of  $X' = (U_+ + U_-) \cap V^\perp$ , we have

$$\begin{aligned} \dim X' &= 2n + 1 - \dim(Z + V) \\ &= 2n + 1 - \dim Z - \dim V + \dim(Z \cap V) \\ &= 2n + 1 - (2a_2 + 1 + d) - n + \dim(Z \cap V) \\ &= n - 2a_2 - d + \dim(Z \cap V) \\ &= a_1 - a_2 + \dim(Z \cap V). \end{aligned}$$

Since  $\pi(X') \cong X'/(W \cap V^\perp)$ ,  $\pi(Z \cap V) \cong (Z \cap V)/(W \cap V)$  and  $W \cap V^\perp = W \cap V$ , we have the desired equality.  $\square$

The bilinear form  $(\ , \ )$  naturally induces a nondegenerate bilinear form on  $W^\perp/W$ . It is nondegenerate on the pair  $(\pi(U_+), \pi(U_-)) \cong (U_+/W_+, U_-/W_-) \cong (U_{(+)}, U_{(-)})$ . Put

$$X_0 = ((U_+ + W_-) \cap V) + ((W_+ + U_-) \cap V).$$

For  $v \in X'$ , write  $v = v_+ + v_-$  with  $v_+ \in U_+ + W_-$  and  $v_- \in W_+ + U_-$ . Then  $\pi(v_+)$  is uniquely defined from  $v$ . Define a subspace

$$X_1 = X_1(V) = \{v \in X' \mid (v_+, X') = \{0\}\}$$

of  $X'$ .

**Lemma 3.9.**  $X_0 \subset X_1 \subset X$ .

*Proof.* Let  $v$  be an element of  $X_0$ . Then  $v = v_+ + v_-$  with  $v_+ \in (U_+ + W_-) \cap V$  and  $v_- \in (W_+ + U_-) \cap V$ . So we have

$$(v_+, X') \subset (V, V^\perp) = \{0\}.$$

Thus we have  $X_0 \subset X_1$ .

Let  $v = v_+ + v_-$  be an element of  $X'$  which is not contained in  $X$ . Then  $(v, v) \neq 0$ . This implies

$$(v_+, v) = (v_+, v_-) \neq 0.$$

Hence  $v \notin X_1$ . Thus we have proved  $X_1 \subset X$ .  $\square$

Put  $b_{12} = \dim X_1 - \dim X_0$ ,  $b_{15} = \dim X' - \dim X_1$  and  $\varepsilon = \dim X' - \dim X \in \{0, 1\}$ .



**Lemma 3.10.** *The projections  $\pi_+$  and  $\pi_-$  induce bijections*

$$X'/X_0 \xrightarrow{\sim} \pi_+(X')/\pi_+(X_0) \quad \text{and} \quad X'/X_0 \xrightarrow{\sim} \pi_-(X')/\pi_-(X_0),$$

*respectively.*

*Proof.* Note that the kernel of the projection  $\pi_+|_{U_+ + U_-} : U_+ + U_- \rightarrow \pi(U_+)$  is  $W_+ + U_-$ . So the kernel of  $\pi_+|_{X'}$  is  $(W_+ + U_-) \cap X' = (W_+ + U_-) \cap V$ . Since  $\pi_+(X_0) = \pi((U_+ + W_-) \cap V^\perp) \cong ((U_+ + W_-) \cap V^\perp)/(W \cap V)$ , we get the first bijection. In the same way, we also get the second one.  $\square$

**Corollary 3.11.**  $b_{12} \leq a_1 - \dim \pi(X')$ .

*Proof.* By Lemma 3.10, we have

$$\dim(\pi_+(X') \oplus \pi_-(X_1)) = \dim \pi(X') + b_{12}.$$

On the other hand,  $\pi_+(X') \oplus \pi_-(X_1)$  is an isotropic subspace of  $\pi(U_+ + U_-) = \pi_+(U_+) \oplus \pi_-(U_-) \cong U_{(+)} \oplus U_{(-)}$  by the definition of  $X_1$ . So we have

$$\dim \pi(X') + b_{12} \leq a_1.$$

$\square$

Put  $b_{13} = a_1 - \dim \pi(X') - b_{12}$  and  $b_{14} = a_2 - b_{12} - b_{13}$ . Then  $b_{14} = \dim \pi(Z \cap V)$  by Lemma 3.8.

By the definition of  $X_1$ , the bilinear form  $(\ , \ )$  is nondegenerate on the pair  $(\pi_+(X')/\pi_+(X_1), \pi_-(X')/\pi_-(X_1))$ . By Lemma 3.10, there exists a bijection

$$f : \pi_+(X')/\pi_+(X_1) \xrightarrow{\sim} \pi_-(X')/\pi_-(X_1)$$

induced by  $\pi_-|_{X'} \circ \pi_+|_{X'}^{-1}$ . Define a bilinear form  $\langle \ , \ \rangle$  on  $\pi_+(X')/\pi_+(X_1)$  by

$$\langle u, v \rangle = (u, f(v)) \quad \text{for } u, v \in \pi_+(X').$$

(This is well-defined since  $(u, f(v)) = 0$  if  $u$  or  $v$  is contained in  $\pi_+(X_1)$ .)

**Lemma 3.12.** *If  $u \in X'$  and  $v \in X$ , then  $\langle \pi_+(u), \pi_+(v) \rangle = -\langle \pi_+(v), \pi_+(u) \rangle$ .*

*Proof.* Write  $u = u_+ + u_-$  and  $v = v_+ + v_-$  with  $u_+, v_+ \in U_+$  and  $u_-, v_- \in U_-$ . Then

$$0 = (u, v) = (u_+ + u_-, v_+ + v_-) = (u_+, v_-) + (u_-, v_+) = \langle \pi_+(u), \pi_+(v) \rangle + \langle \pi_+(v), \pi_+(u) \rangle.$$

$\square$

**Corollary 3.13.**  $\varepsilon = 0 \implies b_{15}$  is even.

Summarizing the arguments in this subsection, we have:

**Proposition 3.14.** *The invariants  $b_1, \dots, b_{15}$  and  $\varepsilon$  for  $V$  satisfy the following equalities.*

$$(3.8) \quad a_0 = b_1 + b_2,$$

$$(3.9) \quad a_+ = b_3 + b_7 + b_8 + b_{10},$$

$$(3.10) \quad a_- = b_4 + b_7 + b_9 + b_{11},$$

$$(3.11) \quad a_1 = b_5 + b_6 + b_8 + b_9 + 2b_{12} + b_{13} + b_{15},$$

$$(3.12) \quad a_2 = b_{12} + b_{13} + b_{14}$$

$$(3.13) \quad \text{and } \varepsilon = \begin{cases} 0 & \text{if } b_{15} = 0, \\ 1 & \text{if } b_{15} \text{ is odd,} \\ 0 \text{ or } 1 & \text{if } b_{15} \text{ is even and positive.} \end{cases}$$

*Proof.* The equality (3.11) follows from the definitions of  $b_{12}, b_{13}, b_{15}$  and

$$\dim \pi(X_0) = b_5 + b_6 + b_8 + b_9.$$

The other equalities follow from the definitions of  $b_1, \dots, b_{15}$  and Corollary 3.13.  $\square$

**3.5. Representative of the  $R$ -orbit of  $V$ .** Conversely suppose that nonnegative numbers  $b_1, \dots, b_{15}$  and  $\varepsilon$  satisfy the equalities in Proposition 3.14. Then we define a maximally isotropic subspace

$$V(b_1, \dots, b_{15}, \varepsilon) = \left( \bigoplus_{j=1}^{14} V_{(j)} \right) \oplus V_{(15)}^\varepsilon$$

as follows.

Define subsets  $I_{(j)}$  of  $I$  for  $j \in J_1 = \{1, 2, 3, 4, 5, 6, 10, 11, 14\}$  by

$$\begin{aligned} I_{(1)} &= \{1, \dots, b_1\}, & I_{(2)} &= \{b_1 + 1, \dots, a_0\}, & I_{(3)} &= \{a_0 + 1, \dots, a_0 + b_3\}, \\ I_{(4)} &= \{a_0 + a_+ + 1, \dots, a_0 + a_+ + b_4\}, & I_{(5)} &= \{d + 1, \dots, d + b_5\}, \\ I_{(6)} &= \{d' + 1, \dots, d' + b_6\}, & I_{(10)} &= \{a_0 + a_+ - b_{10} + 1, \dots, a_0 + a_+\}, \\ I_{(11)} &= \{d - b_{11} + 1, \dots, d\} & \text{and } I_{(14)} &= \{d + a_1 + 1, \dots, d + a_1 + b_{14}\} \end{aligned}$$

where  $d' = \overline{d + a_1} + 1 = 2n + 1 - d - a_1$ . For  $j \in J_1$ , put  $U_{(j)} = \bigoplus_{i \in \widetilde{I_{(j)}}} \mathbb{F}e_i$  where  $\widetilde{I_{(j)}} = I_{(j)} \sqcup \overline{I_{(j)}}$  and define maximally isotropic subspaces  $V_{(j)}$  of  $U_{(j)}$  by

$$V_{(j)} = \begin{cases} \bigoplus_{i \in I_{(j)}} \mathbb{F}e_i & \text{if } j = 1, 3, 4, 5, 6, 14, \\ \bigoplus_{i \in I_{(j)}} \mathbb{F}e_{\bar{i}} & \text{if } j = 2, 10, 11. \end{cases}$$

Define subsets  $I_{(j)}$  of  $I$  and maps  $\eta_j : I_{(j)} \rightarrow I$  for  $j \in J_2 = \{7, 8, 9, 13\}$  by

$$I_{(7)} = \{a_0 + b_3 + 1, \dots, a_0 + b_3 + b_7\},$$

$$\eta_7(a_0 + b_3 + k) = a_0 + a_+ + b_4 + k \quad \text{for } k = 1, \dots, b_7$$

$$I_{(8)} = \{a_0 + b_3 + b_7 + 1, \dots, a_0 + b_3 + b_7 + b_8\},$$

$$\eta_8(a_0 + b_3 + b_7 + k) = d' + b_6 + k \quad \text{for } k = 1, \dots, b_8$$

$$I_{(9)} = \{a_0 + a_+ + b_4 + b_7 + 1, \dots, a_0 + a_+ + b_4 + b_7 + b_9\},$$

$$\eta_9(a_0 + a_+ + b_4 + b_7 + k) = d + b_5 + k \quad \text{for } k = 1, \dots, b_9,$$

$$I_{(13)} = \{d + b_5 + b_9 + b_{12} + b_{15} + 1, \dots, d + b_5 + b_9 + b_{12} + b_{15} + b_{13}\}$$

$$\text{and } \eta_{13}(d + b_5 + b_9 + b_{12} + b_{15} + k) = d + a_1 + b_{14} + b_{12} + k \quad \text{for } k = 1, \dots, b_{13}.$$

For  $j \in J_2$ , put  $U_{(j)} = \bigoplus_{i \in \widetilde{I_{(j)}}} \mathbb{F}e_i$  where  $\widetilde{I_{(j)}} = I_{(j)} \sqcup \eta_j(I_{(j)}) \sqcup \overline{I_{(j)}} \sqcup \overline{\eta_j(I_{(j)})}$ . Define maximally isotropic subspaces  $V_{(j)} = V_{(j)}^1 \oplus V_{(j)}^2$  of  $U_{(j)}$  for  $j \in J_2$  by

$$V_{(j)}^1 = \bigoplus_{i \in I_{(j)}} \mathbb{F}(e_i + e_{\eta_j(i)}) \quad \text{and} \quad V_{(j)}^2 = \bigoplus_{i \in I_{(j)}} \mathbb{F}(e_{\overline{i}} - e_{\overline{\eta_j(i)}}).$$

Define a subset  $I_{(12)} = \{d + b_5 + b_9 + 1, \dots, d + b_5 + b_9 + b_{12}\}$  of  $I$  and maps  $\kappa, \lambda : I_{(12)} \rightarrow I$  given by

$$\kappa(d + b_5 + b_9 + k) = d' + b_6 + b_8 + k, \quad \lambda(d + b_5 + b_9 + k) = d + a_1 + b_{14} + k$$

for  $k = 1, \dots, b_{12}$ . Put  $U_{(12)} = \bigoplus_{i \in \widetilde{I_{(12)}}} \mathbb{F}e_i$  where  $\widetilde{I_{(12)}} = I_{(12)} \sqcup \kappa(I_{(12)}) \sqcup \lambda(I_{(12)}) \sqcup \overline{I_{(12)}} \sqcup \overline{\kappa(I_{(12)})} \sqcup \overline{\lambda(I_{(12)})}$ . Define a maximally isotropic subspace  $V_{(12)} = V_{(12)}^1 \oplus V_{(12)}^2 \oplus V_{(12)}^3$  of  $U_{(12)}$  by

$$V_{(12)}^1 = \bigoplus_{i \in I_{(12)}} \mathbb{F}(e_i + e_{\kappa(i)}), \quad V_{(12)}^2 = \bigoplus_{i \in I_{(12)}} \mathbb{F}(e_i + e_{\lambda(i)})$$

$$\text{and } V_{(12)}^3 = \bigoplus_{i \in I_{(12)}} \mathbb{F}(e_{\overline{i}} - e_{\overline{\kappa(i)}} - e_{\overline{\lambda(i)}}).$$

Put  $U_{(15)} = (\bigoplus_{i \in I_{15} \sqcup \overline{I_{(15)}}} \mathbb{F}e_i) \oplus \mathbb{F}e_{n+1}$  where

$$I_{(15)} = \{d + b_5 + b_9 + b_{12} + 1, \dots, d + b_5 + b_9 + b_{12} + b_{15}\}.$$

Define a map

$$\eta_{15}(d + b_5 + b_9 + b_{12} + k) = d' + b_6 + b_8 + b_{12} + b_{13} + k \quad \text{for } k = 1, \dots, b_{15}.$$

If  $b_{15} = 0$ , then we define  $V_{(15)}^\varepsilon = \{0\}$ . Suppose  $b_{15} > 0$ . Then we put

$$c = d + b_5 + b_9 + b_{12} + \left\lceil \frac{b_{15} + 1}{2} \right\rceil \quad \text{and} \quad I_{(15)}^+ = \{d + b_5 + b_9 + b_{12} + 1, \dots, c\}.$$

For  $i \in I_{(15)}^+ - \{c\}$ , define

$$V_{<i>} = \mathbb{F}(e_i + e_{\eta_{15}(i)}) \oplus \mathbb{F}(e_{\overline{i}} - e_{\overline{\eta_{15}(i)}}).$$

Define

$$V_{<c>}^\varepsilon = \begin{cases} \mathbb{F}(e_c + e_{\eta_{15}(c)}) \oplus \mathbb{F}(e_{\bar{c}} - e_{\overline{\eta_{15}(c)}}) & \text{if } b_{15} \text{ is even and } \varepsilon = 0, \\ \mathbb{F}(e_c + e_{\eta_{15}(c)}) \oplus \mathbb{F}(e_{\bar{c}} - e_{\overline{\eta_{15}(c)}} - \frac{1}{2}e_c + e_{n+1}) & \text{if } b_{15} \text{ is even and } \varepsilon = 1, \\ \mathbb{F}(e_{\bar{c}} - \frac{1}{2}e_c + e_{n+1}) & \text{if } b_{15} \text{ is odd } (\varepsilon = 1). \end{cases}$$

Then we define maximally isotropic subspaces  $V_{(15)}^\varepsilon$  of  $U_{(15)}$  by

$$V_{(15)}^\varepsilon = \left( \bigoplus_{i \in I_{(15)}^+ - \{c\}} V_{<i>}^\varepsilon \right) \oplus V_{<c>}^\varepsilon.$$

**Theorem 3.15.** *Let  $V$  be a maximally isotropic subspace of  $\mathbb{F}^{2n+1}$ . Define the numbers  $b_1, \dots, b_{15}$  and  $\varepsilon$  as in Section 3.4. Then the  $R$ -orbit of  $V$  contains the representative*

$$V(b_1, \dots, b_{15}, \varepsilon).$$

*Remark 3.16.* If  $V = V(b_1, \dots, b_{15}, \varepsilon)$ , then we can write for  $S = V, V^\perp, U_+, U_+^\perp, U_-$  and  $U_-^\perp$ ,

$$S = \left( \bigoplus_{i \in I_{(1)} \sqcup \dots \sqcup I_{(14)}} ((W_{<i>} \cap S) \oplus (\overline{W}_{<i>} \cap S)) \right) \oplus (U_{(15)} \cap S)$$

where  $W_{<i>} = \bigoplus_{k \in I_{<i>}} \mathbb{F}e_k$  and  $\overline{W}_{<i>} = \bigoplus_{k \in I_{<i>}} \mathbb{F}e_{\bar{k}}$  with

$$I_{<i>} = \begin{cases} \{i\} & \text{if } i \in I_{(j)} \text{ with } j \in J_1, \\ \{i, \eta_j(i)\} & \text{if } i \in I_{(j)} \text{ with } j \in J_2, \\ \{i, \kappa(i), \lambda(i)\} & \text{if } i \in I_{(12)}. \end{cases}$$

So the subspaces  $W_{<i>} \oplus \overline{W}_{<i>}$  are “indecomposable summands” in the sense of [MWZ00]. They are characterized by

$$\mathbf{d}(W) = ((d_1^V(W), d_2^V(W), d_3^V(W)), (d_1^+(W), d_2^+(W), d_3^+(W)), (d_1^-(W), d_2^-(W), d_3^-(W)))$$

for  $W = W_{<i>}$  and  $W = \overline{W}_{<i>}$  where

$$\begin{aligned} d_1^V(W) &= \dim(W \cap V), \\ d_2^V(W) &= \dim(W \cap V^\perp) - \dim(W \cap V), \\ d_3^V(W) &= \dim W - \dim(W \cap V^\perp), \\ d_1^+(W) &= \dim(W \cap U_+), \\ d_2^+(W) &= \dim(W \cap U_+^\perp) - \dim(W \cap U_+), \\ d_3^+(W) &= \dim W - \dim(W \cap U_+^\perp), \\ d_1^-(W) &= \dim(W \cap U_-), \\ d_2^-(W) &= \dim(W \cap U_-^\perp) - \dim(W \cap U_-), \\ \text{and } d_3^-(W) &= \dim W - \dim(W \cap U_-^\perp). \end{aligned}$$

$i \in I_{(j)}$	$\mathbf{d}(W_{<i>})$	$\mathbf{d}(\overline{W}_{<i>})$
$I_{(1)}$	$((10)(10)(10))$	$((01)(01)(01))$
$I_{(2)}$	$((01)(10)(10))$	$((10)(01)(01))$
$I_{(3)}$	$(10)(10)(1)$	$((01)(01)(1))$
$I_{(4)}$	$((10)(1)(10))$	$((01)(1)(01))$
$I_{(5)}$	$((10)(10)(01))$	$((01)(01)(10))$
$I_{(6)}$	$((10)(01)(10))$	$((01)(10)(01))$
$I_{(7)}$	$((11)(110)(110))$	$((11)(011)(011))$
$I_{(8)}$	$((11)(11)(110))$	$((11)(11)(011))$
$I_{(9)}$	$((11)(110)(11))$	$((11)(011)(11))$
$I_{(10)}$	$(01)(10)(1)$	$((10)(01)(1))$
$I_{(11)}$	$((01)(1)(10))$	$((10)(1)(01))$
$I_{(12)}$	$((21)(111)(111))$	$((12)(111)(111))$
$I_{(13)}$	$((11)(110)(011))$	$((01)(011)(110))$
$I_{(14)}$	$((10)(1)(1))$	$((01)(1)(1))$

 TABLE 1.  $\mathbf{d}(W_{<i>})$  and  $\mathbf{d}(\overline{W}_{<i>})$ 

For  $i \in I_{(j)}$  with  $j = 1, \dots, 14$ ,  $\mathbf{d}(W_{<i>})$  and  $\mathbf{d}(\overline{W}_{<i>})$  are as in Table 1. Note that

$$d_k^*(W_{<i>}) = d_{4-k}^*(\overline{W}_{<i>})$$

for  $*$  =  $V, +, -$  and  $k = 1, 2, 3$ . We write them in the “compressed form” as in [MWZ00]: We omit  $d_k^*(W)$  and  $d_{4-k}^*(W)$  if both are 0. We also omit commas.

We can also decompose  $U_{(15)}$  into indecomposable summands.

First suppose that  $\varepsilon = 0$ . (Then  $b_{15}$  is even.) Put  $W_{<i>}^0 = \mathbb{F}e_i \oplus \mathbb{F}e_{\eta_{15}(i)}$ ,  $\overline{W}_{<i>}^0 = \mathbb{F}e_{\bar{i}} \oplus \mathbb{F}e_{\overline{\eta_{15}(i)}}$  for  $i \in I_{(15)}^+$  and  $W_{<n+1>} = \mathbb{F}e_{n+1}$ . Then

$$U_{(15)} = \left( \bigoplus_{i \in I_{(15)}^+} (W_{<i>}^0 \oplus \overline{W}_{<i>}^0) \right) \oplus W_{<n+1>},$$

$\mathbf{d}(W_{<i>}^0) = \mathbf{d}(\overline{W}_{<i>}^0) = ((11)(11)(11))$  and  $\mathbf{d}(W_{<n+1>}) = ((1)(1)(1))$ .

Next suppose that  $b_{15}$  is even and  $\varepsilon = 1$ . Put  $W_{<c>}^{\text{even}} = \mathbb{F}e_c \oplus \mathbb{F}e_{\eta_{15}(c)} \oplus \mathbb{F}e_{\bar{c}} \oplus \mathbb{F}e_{\overline{\eta_{15}(c)}}$  for  $c \in I_{(15)}^+ - \{c\}$ . Then

$$U_{(15)} = \left( \bigoplus_{i \in I_{(15)}^+ - \{c\}} (W_{<i>}^0 \oplus \overline{W}_{<i>}^0) \right) \oplus W_{<c>}^{\text{even}}$$

and  $\mathbf{d}(W_{<c>}^{\text{even}}) = ((212)(212)(212))$ .

Finally suppose that  $b_{15}$  is odd. (Then  $\varepsilon = 1$ .) Put  $W_{<c>}^{\text{odd}} = \mathbb{F}e_c \oplus \mathbb{F}e_{\bar{c}} \oplus \mathbb{F}e_{n+1}$ . Then

$$U_{(15)} = \left( \bigoplus_{i \in I_{(15)}^+ - \{c\}} (W_{<i>}^0 \oplus \overline{W}_{<i>}^0) \right) \oplus W_{<c>}^{\text{odd}}$$

and  $\mathbf{d}(W_{<c>}^{\text{odd}}) = ((111)(111)(111))$ .

**3.6. Proof of Theorem 3.15.** (i)  $V_{(1)}$ -part: There exists an element  $g \in R$  of the form

$$g = \ell \left( \begin{pmatrix} A & 0 \\ 0 & I_{a_+ + a_-} \end{pmatrix} \right)$$

with  $A \in \text{GL}_{a_0}(\mathbb{F})$  such that

$$g(W_0 \cap V) = V_{(1)} = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_1}.$$

Put  $V'_1 = gV$  and let  $U_1$  be the orthogonal complement of  $U_{(1)} = V_{(1)} \oplus \overline{V}_{(1)}$  in  $\mathbb{F}^{2n+1}$  where  $\overline{V}_{(1)} = \bigoplus_{i \in I_{(1)}} \mathbb{F}e_{\bar{i}}$ . Then

$$V'_1 = gV = V_{(1)} \oplus V_1.$$

with  $V_1 = U_1 \cap V'_1$ . So we have only to consider the  $(O(U_1) \cap R)$ -orbit of  $V_1$  in the following. Since  $gV \cap W_0 = g(V \cap W_0) = V_{(1)}$ , we have

$$V_1 \cap W_0 = \{0\}.$$

(ii)  $V_{(2)}$ -part: Let  $U_2$  denote the orthogonal complement of  $U_{(2)} = \overline{V}_{(2)} \oplus V_{(2)}$  in  $U_1$  where  $\overline{V}_{(2)} = \bigoplus_{i \in I_{(2)}} \mathbb{F}e_i$ . Then  $U_2 = \mathbb{F}e_{a_0+1} \oplus \cdots \oplus \mathbb{F}e_{\overline{a_0+1}}$  and

$$U_1 = \overline{V}_{(2)} \oplus U_2 \oplus V_{(2)}.$$

Let  $p$  denote the projection of  $U_1$  onto  $V_{(2)}$  with respect to this direct sum decomposition. Since  $V_1 \cap \overline{V}_{(2)} = \{0\}$ , we have  $p(V_1) = V_{(2)}$  by Lemma 3.4 (i). So we can write

$$V_1 = (V_1 \cap (\overline{V}_{(2)} \oplus U_2)) \oplus \bigoplus_{j \in \overline{I_{(2)}}} \mathbb{F}v_j$$

with vectors  $v_j \in e_j + (\overline{V}_{(2)} \oplus U_2)$  for  $j \in \overline{I_{(2)}}$ . By Lemma 3.5, we can take a  $g_1 \in O(U_1)$  such that

$$g_1 e_j \begin{cases} = e_j & \text{if } j \in I_{(2)}, \\ = v_j & \text{if } j \in \overline{I_{(2)}}, \\ \in e_j + \overline{V}_{(2)} & \text{if } j \in I - I_{(2)} - \overline{I_{(2)}}. \end{cases}$$

Since  $\overline{V}_{(2)} = \bigoplus_{i \in I_{(2)}} \mathbb{F}e_i \subset W_0$ , we have  $g_1 U_{\pm} = U_{\pm}$  and hence  $g_1 \in R$ . Since  $g_1^{-1}V_1 \supset V_{(2)}$ , we have  $g_1^{-1}V_1 \subset U_2 \oplus V_{(2)}$  and hence

$$g_1^{-1}V_1 = (g_1^{-1}V_1 \cap U_2) \oplus V_{(2)} = V_2 \oplus V_{(2)}$$

where  $V_2 = g_1^{-1}V_1 \cap U_2$ . So we have only to consider  $(O(U_2) \cap R)$ -orbit of  $V_2$  in the following since  $V'_2 = g_1^{-1}V'_1$  is written as  $V'_2 = V_{(1)} \oplus V_{(2)} \oplus V_2$ .

(iii)  $V_{(3)}$  and  $V_{(4)}$ -parts: We can take an element

$$g_2 = \ell \left( \begin{pmatrix} I_{a_0} & & 0 \\ & A & \\ 0 & & B \end{pmatrix} \right) \in O(U_2) \cap R \cap L_W$$

with some  $A \in \mathrm{GL}_{a_+}(\mathbb{F})$  and  $B \in \mathrm{GL}_{a_-}(\mathbb{F})$  such that

$$g_2(W_+ \cap V_2) = V_{(3)} \quad \text{and} \quad g_2(W_- \cap V_2) = V_{(4)}.$$

Let  $U_4$  denote the orthogonal complement of  $U_{(3)} \oplus U_{(4)}$  in  $U_2$ . Then we have

$$g_2 V_2 = V_{(3)} \oplus V_{(4)} \oplus V_4$$

with  $V_4 = g_2 V_2 \cap U_4$ . So we have only to consider the  $(O(U_4) \cap R)$ -orbit of  $V_4$  in the following since  $V'_4 = g_2 V'_2$  is written as  $V'_4 = V_{(1)} \oplus V_{(2)} \oplus V_{(3)} \oplus V_{(4)} \oplus V_4$ . Note that

$$(3.14) \quad V_4 \cap W_+ = V_4 \cap W_- = \{0\}.$$

(iv)  $V_{(5)}$  and  $V_{(6)}$ -parts: Let  $p : W \oplus U \rightarrow U$  denote the canonical projection map. (Note that  $p$  is the composition of  $\pi : W^\perp = W \oplus U \rightarrow W^\perp/W$  and the identification  $W^\perp/W \cong U$ .) It follows from (3.14) that  $\dim p(V_4 \cap U_+) = b_5$  and that  $\dim p(V_4 \cap U_-) = b_6$ . Since  $(p(V_4 \cap U_+), p(V_4 \cap U_-)) = \{0\}$ , we have

$$p(V_4 \cap U_+) \subset U_{(+)}^{\perp p(V_4 \cap U_-)}.$$

Hence we can take an element

$$g_3 = \ell_{00}(A, I_{m-2a_1}) \in R \cap L_U$$

with some  $A \in \mathrm{GL}_{a_1}(\mathbb{F})$  such that

$$g_3 p(V_4 \cap U_+) = V_{(5)}$$

$$\text{and that } g_3 U_{(+)}^{\perp p(V_4 \cap U_-)} = U_{(+)}^{\perp V_{(6)}} \quad (\iff g_3 p(V_4 \cap U_-) = V_{(6)}).$$

We can write  $V_4 \cap U_+ = \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_{b_5}$  and  $V_4 \cap U_- = \mathbb{F}v'_1 \oplus \cdots \oplus \mathbb{F}v'_{b_6}$  with vectors

$$v_j = e_{d+j} + \sum_{i=a_0+b_3+1}^{a_0+a_+} y_{i,j} e_i$$

for  $j = 1, \dots, b_5$  and

$$v'_j = e_{d'+j} + \sum_{i=a_0+a_++b_4+1}^d y'_{i,j} e_i$$

for  $j = 1, \dots, b_6$  with some matrices  $\{y_{i,j}\}$  and  $\{y'_{i,j}\}$ . Define a matrix  $X = \{x_{i,j}\}_{j=1, \dots, m}^{i=1, \dots, d}$  by

$$x_{i,j} = \begin{cases} y_{i,j} & \text{if } (i, j) \in \{a_0 + b_3 + 1, \dots, a_0 + a_+\} \times \{1, \dots, b_5\}, \\ y'_{i, d'-d+j} & \text{if } (i, j) \in \{a_0 + a_+ + b_4 + 1, \dots, d\} \times \{d' - d + 1, \dots, d' - d + b_6\}, \\ 0 & \text{otherwise} \end{cases}$$

and put  $g_4 = g(X, 0) \in O(U_4) \cap R \cap N_W$  (Proposition 3.7 (iv)). Then we have

$$g_4^{-1}v_j = e_{d+j} \quad \text{for } j = 1, \dots, b_5 \quad \text{and} \quad g_4^{-1}v'_j = e_{d'+j} \quad \text{for } j = 1, \dots, b_6.$$

Thus we have

$$g_4^{-1}g_3(V_4 \cap U_+) = V_{(5)} \quad \text{and} \quad g_4^{-1}g_3(V_4 \cap U_-) = V_{(6)}.$$

Take the orthogonal complement  $U_6$  of  $U_{(5)} \oplus U_{(6)}$  in  $U_4$ . Then we have

$$g_4^{-1}g_3V_4 = V_{(5)} \oplus V_{(6)} \oplus V_6$$

with  $V_6 = g_4^{-1}g_3V_4 \cap U_6$ . So we have only to consider the  $(O(U_6) \cap R)$ -orbit of  $V_6$  in the following since  $V'_6 = g_4^{-1}g_3V'_4$  is written as  $V'_6 = (\bigoplus_{j=1}^6 V_{(j)}) \oplus V_6$ . Note that

$$(3.15) \quad V_6 \cap U_+ = V_6 \cap U_- = \{0\}.$$

Let  $p_{\pm} : U_6 \cap (U_+ + U_-) = (U_6 \cap U_+) \oplus (U_6 \cap U_-) \rightarrow U_6 \cap U_{\pm}$  denote the projections with respect to the direct sum decomposition. By (3.15), we have:

**Proposition 3.17.** *The projections give injections*

$$p_{\pm} : V_6 \cap (U_+ + U_-) \rightarrow U_6 \cap U_{\pm}.$$

(v)  $V_{(7)}$ -part: Let  $U_7$  denote the orthogonal complement of  $U_{(7)}$  in  $U_6$ . Then we can write  $U_7 = \bigoplus_{i \in I_7} \mathbb{F}e_i$  with  $I_7 = I_6 - \widetilde{I_{(7)}} = I - (\bigsqcup_{j=1}^7 \widetilde{I_{(j)}})$ .

By Proposition 3.17, we can take an element

$$g_5 = \ell \left( \begin{pmatrix} I_{a_0+b_3} & & 0 \\ & A & \\ 0 & & I_{b_4} & \\ & & & B \end{pmatrix} \right) \in O(U_6) \cap R \cap L_W$$

with some  $A \in \text{GL}_{a_+-b_3}(\mathbb{F})$  and  $B \in \text{GL}_{a_--b_4}(\mathbb{F})$  such that

$$g_5V_6 \cap W = V_{(7)}^1.$$

Since  $\overline{W} \cap U_6 = \bigoplus_{i \in I_{(7)} \sqcup I_{(8)} \sqcup I_{(9)} \sqcup I_{(10)} \sqcup I_{(11)}} \mathbb{F}e_{\bar{i}}$ , it follows that

$$(3.16) \quad p_{\overline{W}}(g_5V_6) = V_{(7)}^2 \oplus \bigoplus_{i \in I_{(8)} \sqcup I_{(9)} \sqcup I_{(10)} \sqcup I_{(11)}} \mathbb{F}e_{\bar{i}}$$

by Lemma 3.4 (i).

By (3.16) we can write

$$g_5V_6 = ((V_{(7)}^1 \oplus U_7) \cap g_5V_6) \oplus \bigoplus_{j \in I_{(7)}} \mathbb{F}v_j$$

with vectors

$$v_j = e_{\bar{j}} - e_{\overline{\eta(j)}} + \sum_{i \in I_6 \cap I_{W^\perp}} c_{i,j} e_i$$

for  $j \in I_{(7)}$  with  $c_{i,j} \in \mathbb{F}$  where  $I_{W^\perp} = \{i \in I \mid e_i \in W^\perp\}$ .



Put

$$u_j = -e_{\overline{\eta_7(j)}} + \sum_{i \in I_{U_{(+)}} \cap I_6} c_{i,j} e_i \quad \text{and} \quad v_j^+ = u_j - \sum_{i \in I_{(7)}} (u_j, v_i) e_i$$

for  $j \in I_{(7)}$  where  $I_{U_{(+)}} = \{i \in I \mid e_i \in U_{(+)}\} = \{d+1, \dots, d+a_1\}$ . Then  $(v_j^+, v_k^+) = 0$  and

$$(v_j^+, v_k) = (u_j, v_k) - \sum_{i \in I_{(7)}} (u_j, v_i) (e_i, v_k) = (u_j, v_k) - (u_j, v_k) = 0$$

for  $j, k \in I_{(7)}$ . Write  $v_j^- = v_j - v_j^+$ . Then we can take an element  $g_6 \in O(U_6)$  such that

$$g_6 e_j = e_j, \quad g_6 e_{\eta_7(j)} = e_{\eta_7(j)}, \quad g_6 e_{\overline{j}} = v_j^- \quad \text{and that} \quad -g_6 e_{\overline{\eta_7(j)}} = v_j^+$$

for  $j \in I_{(7)}$  since  $(v_j^+, v_k^+) = (v_j^-, v_k^-) = (v_j^+, v_k^-) = 0$  for  $j, k \in I_{(7)}$  by Lemma 3.5. The element  $g_6$  is also contained in  $R \cap N_W$  by its construction. Hence we have

$$g_6^{-1} g_5 V_6 = V_{(7)}^1 \oplus (U_7 \cap g_6^{-1} g_5 V_6) \oplus V_{(7)}^2.$$

So we have only to consider the  $(O(U_7) \cap R)$ -orbit of  $V_7 = U_7 \cap g_6^{-1} g_5 V_6$  in the following since  $V_7' = g_6^{-1} g_5 V_6'$  is written as  $V_7' = (\bigoplus_{j=1}^7 V_{(j)}) \oplus V_7$ .

(vi)  $V_{(8)}$ -part: Let  $U_8$  denote the orthogonal complement of  $U_{(8)}$  in  $U_7$ . Then we can write  $U_8 = \bigoplus_{i \in I_8} \mathbb{F} e_i$  with  $I_8 = I_7 - \widetilde{I_{(8)}}$ .

By Proposition 3.17,  $\dim p_+((W_+ + U_-) \cap V_7) = \dim p_-((W_+ + U_-) \cap V_7) = b_8$ . Furthermore  $p_-((W_+ + U_-) \cap V_7) \cap W_- = \{0\}$  by the definition of  $V_7$ . So we can take elements

$$g_7 = \ell \left( \begin{pmatrix} I_{a_0+b_3+b_7} & 0 \\ & A \\ 0 & & I_{a_-} \end{pmatrix} \right) \in O(U_7) \cap R \cap L_W$$

and  $g_8 = \ell_{00} \left( \begin{pmatrix} I_{b_5} & 0 \\ & B \\ 0 & & I_{b_6} \end{pmatrix}, I_{m-2a_1} \right) \in O(U_7) \cap R \cap L_U$

with  $A \in \text{GL}_{a_+-b_3-b_7}(\mathbb{F})$  and  $B \in \text{GL}_{a_1-b_5-b_6}(\mathbb{F})$  such that

$$g_8 g_7 ((W_+ + U_-) \cap V_7) = \bigoplus_{j \in I_{(8)}} \mathbb{F}(e_j + w_j)$$

where  $w_j \in e_{\eta_8(j)} + (W_- \cap V_7)$ . By Proposition 3.7 (iv), we can take an element  $g_9 \in \text{SO}(U_7) \cap R \cap N_W$  such that  $g_9 e_{\eta_8(j)} = w_j$  for  $j = 1, \dots, b_8$ . Thus we have

$$g_9^{-1} g_8 g_7 ((W_+ + U_-) \cap V_7) = V_{(8)}^1.$$

Since  $g_9^{-1} g_8 g_7 V_7 \cap W = \{0\}$ , it follows from Lemma 3.4 that  $p_{\overline{W}}(g_9^{-1} g_8 g_7 V_7) = U_7 \cap \overline{W} = \bigoplus_{i \in I_{(8)} \sqcup I_{(9)} \sqcup I_{(10)} \sqcup I_{(11)}} \mathbb{F} e_{\overline{i}}$ . So we can take vectors  $v_j$  in  $g_9^{-1} g_8 g_7 V_7$  for  $j \in I_{(8)}$  of the form

$$v_j = e_{\overline{j}} + \sum_{i \in I_{(7)} \cap I_{W^\perp}} c_{i,j} e_i.$$

Since  $g_9^{-1}g_8g_7V_7$  contains  $V_{(8)}^1$ , they are of the form

$$v_j = e_{\overline{j}} - e_{\overline{\eta_8(j)}} + \sum_{i \in (I_7 \cap I_{W^\perp}) - \overline{\eta_8(I_{(8)})}} c_{i,j} e_i.$$

Define vectors

$$u_j = -e_{\overline{\eta_8(j)}} + \sum_{i \in (I_7 \cap I_{U_{(+)})} - \overline{\eta_8(I_{(8)})}} c_{i,j} e_i \quad \text{and} \quad v_j^+ = u_j - \sum_{i \in I_{(8)}} (u_j, v_i) e_i$$

for  $j \in I_{(8)}$ . Then we have  $(v_j^+, v_k^+) = 0$  and

$$(v_j^+, v_k) = (u_j, v_k) - \sum_{i \in I_{(8)}} (u_j, v_i)(e_i, v_k) = (u_j, v_k) - (u_j, v_k) = 0$$

for  $j, k \in I_{(8)}$ . Put  $v_j^- = v_j - v_j^+$ . Since  $(v_j^+, v_k^+) = (v_j^-, v_k^-) = (v_j^+, v_k^-) = 0$  for  $j, k \in I_{(8)}$ , we can take an element  $g_{10} \in O(U_7)$  such that

$$g_{10}e_j = e_j, \quad g_{10}e_{\eta_8(j)} = e_{\eta_8(j)}, \quad g_{10}e_{\overline{\eta_8(j)}} = -v_j^+ \quad \text{and that} \quad g_{10}e_{\overline{j}} = v_j^-$$

by Lemma 3.5. The element  $g_{10}$  is also contained in  $R \cap N_W$  by its construction. Hence we have

$$g_{10}^{-1}g_9^{-1}g_8g_7V_7 = V_{(8)}^1 \oplus (U_8 \cap g_{10}^{-1}g_9^{-1}g_8g_7V_7) \oplus V_{(8)}^2.$$

So we have only to consider the  $(O(U_8) \cap R)$ -orbit of  $V_8 = U_8 \cap g_{10}^{-1}g_9^{-1}g_8g_7V_7$  in the following since  $V_8' = g_{10}^{-1}g_9^{-1}g_8g_7V_7'$  is written as  $V_8' = (\bigoplus_{j=1}^8 V_{(j)}) \oplus V_8$ .

(vii)  $V_{(9)}$ -part: Let  $U_9$  denote the orthogonal complement of  $U_{(9)}$  in  $U_8$ . Then we can write  $U_9 = \bigoplus_{i \in I_9} \mathbb{F}e_i$  with  $I_9 = I_8 - \widetilde{I_{(9)}}$ . We can take an element  $g' \in O(U_8) \cap R$  such that

$$g'V_8 = V_{(9)}^1 \oplus (g'V_8 \cap U_9) \oplus V_{(9)}^2$$

in the same way as in (vi). So we have only to consider the  $(O(U_9) \cap R)$ -orbit of  $V_9 = U_9 \cap g'V_8$  since  $V_9' = g'V_8'$  is written as  $V_9' = (\bigoplus_{j=1}^9 V_{(j)}) \oplus V_9$ .

(viii)  $V_{(10)}$  and  $V_{(11)}$ -parts: Considering  $V_9'$  instead of  $V$ , we may define

$$X = X(V_9') = (U_+ + U_-) \cap V_9', \quad X' = X'(V_9') = (U_+ + U_-) \cap (V_9')^\perp$$

$$\text{and} \quad X_0 = X_0(V_9') = ((W_+ + U_-) \cap V_9') + ((U_+ + W_-) \cap V_9').$$

Since  $V_9' = V_{(1)} \oplus \cdots \oplus V_{(9)} \oplus V_9$ , we have

$$(V_9')^\perp = V_{(1)} \oplus \cdots \oplus V_{(9)} \oplus (U_9 \cap V_9^\perp).$$

We see that

$$X_0 = V_{(1)} \oplus V_{(3)} \oplus V_{(4)} \oplus V_{(5)} \oplus V_{(6)} \oplus V_{(7)}^1 \oplus V_{(8)}^1 \oplus V_{(9)}^1.$$

So it follows from Lemma 3.10 that the maps

$$\pi_+|_{U_9 \cap X'} : U_9 \cap X' \rightarrow \pi_+(U_9 \cap X') \subset U_{(+)}$$

$$\text{and} \quad \pi_-|_{U_9 \cap X'} : U_9 \cap X' \rightarrow \pi_-(U_9 \cap X') \subset U_{(-)}$$

are bijective. Here we identify  $U_{\pm}/W_{\pm}$  with  $U_{(\pm)}$  and consider the image of  $\pi_{\pm}|_{U_9 \cap X'}$  as subspaces of  $U_{(\pm)}$ . There exist linear maps

$$\varphi_+ : U_9 \cap X' \rightarrow W_+ \cap U_9 = \overline{V}_{(10)} \quad \text{and} \quad \varphi_- : U_9 \cap X' \rightarrow W_- \cap U_9 = \overline{V}_{(11)}$$

such that

$$v = \varphi_+(v) + \varphi_-(v) + \pi_+(v) + \pi_-(v) \quad \text{for } v \in U_9 \cap X'.$$

Consider the maps

$$\widetilde{\varphi}_{\pm} = \varphi_{\pm} \circ (\pi_{\pm}|_{U_9 \cap X'})^{-1} : \pi_{\pm}(U_9 \cap X') \rightarrow U_9 \cap W_{(\pm)}$$

and extend them linearly to  $U_9 \cap U_{(\pm)}$ . Then we can take an element  $h \in \text{O}(U_9) \cap R \cap N_W$  such that

$$h(u) = u + \widetilde{\varphi}_{\pm}(u)$$

for  $u \in U_9 \cap U_{(\pm)}$  by Proposition 3.7 (iv). For  $v \in U_9 \cap X'$ , we have

$$\begin{aligned} h^{-1}v &= h^{-1}(\varphi_+(v) + \pi_+(v) + \varphi_-(v) + \pi_-(v)) \\ &= h^{-1}(\widetilde{\varphi}_+\pi_+(v) + \pi_+(v) + \widetilde{\varphi}_-\pi_-(v) + \pi_-(v)) \\ &= h^{-1}(h\pi_+(v) + h\pi_-(v)) = \pi_+(v) + \pi_-(v) \in U_9 \cap U. \end{aligned}$$

Hence  $h^{-1}(U_9 \cap X') \subset U$ . This implies

$$(e_i, h^{-1}(U_9 \cap X')) = \{0\} \quad \text{for } i \in \overline{I}_{(10)} \sqcup \overline{I}_{(11)}.$$

Since the orthogonal space of  $h^{-1}(U_9 \cap X') = U_9 \cap h^{-1}X' = U_9 \cap (U_+ + U_-) \cap h^{-1}V_9^{\perp}$  in  $U_9$  is  $(U_9 \cap Z) + (U_9 \cap h^{-1}V_9)$ , we can write

$$e_i = v_i - z_i$$

with  $v_i \in U_9 \cap h^{-1}V_9$  and  $z_i \in U_9 \cap Z$  for  $i \in \overline{I}_{(10)} \sqcup \overline{I}_{(11)}$ . By Lemma 3.5, we can take an element  $h_2 \in \text{SO}(U_9) \cap R \cap N_W$  such that

$$h_2 e_i = e_i + z_i = v_i$$

for  $i \in \overline{I}_{(10)} \sqcup \overline{I}_{(11)}$ . So we have

$$h_2^{-1}h^{-1}V_9 \supset V_{(10)} \oplus V_{(11)}.$$

Let  $U_{11}$  be the orthogonal complement of  $\overline{V}_{(10)} \oplus \overline{V}_{(11)} \oplus V_{(10)} \oplus V_{(11)} = U_9 \cap (W \oplus \overline{W})$  in  $U_9$ . Then

$$h_2^{-1}h^{-1}V_9 = (h_2^{-1}h^{-1}V_9 \cap U_{11}) \oplus V_{(10)} \oplus V_{(11)}.$$

Put  $V_{11} = h_2^{-1}h^{-1}V_9 \cap U_{11}$  and  $V'_{11} = h_2^{-1}h^{-1}V'_9$ . Then  $V'_{11} = (\bigoplus_{j=1}^{11} V_{(j)}) \oplus V_{11}$ .

(ix)  $V_{(12)}, V_{(13)}, V_{(14)}$  and  $V_{(15)}^{\varepsilon}$ -parts: Put  $I_{11} = I_9 - (\widetilde{I}_{(10)} \sqcup \widetilde{I}_{(11)}) = I - \bigsqcup_{j=1}^{11} \widetilde{I}_{(j)}$ . Then

$$\begin{aligned} I_{11} &= \{d + b_5 + b_9 + 1, \dots, d + a_1 - b_6 - b_8, d + a_1 + 1, \dots, \overline{d + a_1 + 1}, \\ &\quad \overline{d + a_1 - b_6 - b_8}, \dots, \overline{d + b_5 + b_9 + 1}\}. \end{aligned}$$

With respect to the basis  $\{e_i \mid i \in I_{11}\}$  of  $U_{11}$ , every element of  $O(U_{11}) \cap R$  is represented by a matrix of the form

$$\ell(A, B) = \begin{pmatrix} A & & 0 \\ & B & \\ 0 & & J_{a'_1} {}^t A^{-1} J_{a'_1} \end{pmatrix}$$

with  $A \in \mathrm{GL}_{a'_1}(\mathbb{F})$  and  $B \in \mathrm{O}_{2a_2+1}(\mathbb{F})$  where  $a'_1 = a_1 - b_5 - b_6 - b_8 - b_9$ .

Write  $Y = U_{11} \cap X(V'_{11})$ ,  $Y' = U_{11} \cap X'(V'_{11})$  and  $Y_1 = U_{11} \cap X^1(V'_{11})$ . Then

$$\dim Y' = b_{12} + b_{15}, \quad \dim Y_1 = b_{12} \quad \text{and} \quad \dim Y' - \dim Y = \varepsilon.$$

As is shown in (viii), the maps  $\pi_{\pm}|_{Y'} : Y' \rightarrow \pi_{\pm}(Y') \subset U_{11} \cap U_{\pm}$  are bijective.

Put

$$\begin{aligned} U_{(12)}^{+1} &= \bigoplus_{i \in I_{(12)}} \mathbb{F}e_i, & U_{(12)}^{-2} &= \bigoplus_{i \in I_{(12)}} \mathbb{F}e_{\kappa(i)}, & U_{(13)}^{+} &= \bigoplus_{i \in I_{(13)}} \mathbb{F}e_i, \\ U_{(15)}^{+} &= \bigoplus_{i \in I_{(15)}} \mathbb{F}e_i & \text{and} & & U_{(15)}^{-} &= \bigoplus_{i \in I_{(15)}} \mathbb{F}e_{\bar{i}}. \end{aligned}$$

Note that the space  $\pi_{-}(Y')$  is determined by its orthogonal space  $(U_{11} \cap U_{+})^{\perp \pi_{-}(Y')}$  in  $U_{11} \cap U_{+}$ . Since  $\dim \pi_{+}(Y') = b_{12} + b_{15}$ ,  $\dim (U_{11} \cap U_{+})^{\perp \pi_{-}(Y')} = b_{12} + b_{13}$  and since the dimension of  $\pi_{+}(Y') \cap (U_{11} \cap U_{+})^{\perp \pi_{-}(Y')} = \pi_{+}(Y_1)$  is  $b_{12}$ , we can take an element  $\ell_1 = \ell(A, I_{2a_2+1})$  with some  $A \in \mathrm{GL}_{a'_1}(\mathbb{F})$  such that

$$\begin{aligned} \pi_{+}(\ell_1 Y') &= \ell_1 \pi_{+}(Y') = U_{(12)}^{+1} \oplus U_{(15)}^{+}, \\ (U_{11} \cap U_{+})^{\perp \pi_{-}(\ell_1 Y')} &= \ell_1 (U_{11} \cap U_{+})^{\perp \pi_{-}(Y')} = U_{(12)}^{+1} \oplus U_{(13)}^{+} \\ \text{and that } \pi_{+}(\ell_1 Y_1) &= \ell_1 \pi_{+}(Y_1) = U_{(12)}^{+1} \end{aligned}$$

by Lemma 6.1 in the appendix. The second formula implies  $\pi_{-}(\ell_1 Y') = U_{(12)}^{-2} \oplus U_{(15)}^{-}$ .

Consider the bijective linear map

$$f = \pi_{+}|_{\ell_1 Y'} \circ (\pi_{-}|_{\ell_1 Y'})^{-1} : U_{(12)}^{-2} \oplus U_{(15)}^{-} \rightarrow U_{(12)}^{+1} \oplus U_{(15)}^{+}.$$

Since  $\pi_{-}(\ell_1 Y_1) = U_{(12)}^{-2}$ , we have  $f(U_{(12)}^{-2}) = U_{(12)}^{+1}$ . Write  $f(e_{\kappa(i)}) = \sum_{j \in I_{(12)}} a_{i,j} e_j$  for  $i \in I_{(12)}$ . Define  $\ell_2 = \ell(A, I_{2a_2+1}) \in O(U_{11}) \cap R$  with the matrix

$$A = \begin{pmatrix} \{a_{i,j}\} & 0 \\ 0 & I_{a'_1 - b_{12}} \end{pmatrix}.$$

Then  $\ell_2^{-1} \ell_1 Y_1 = V_{(12)}^1 = \bigoplus_{i \in I_{(12)}} \mathbb{F}(e_i + e_{\kappa(i)})$ . We can write  $f(U_{(15)}^{-}) = \bigoplus_{i \in I_{(15)}} \mathbb{F}v_i$  with vectors  $v_i = e_i + \sum_{j \in I_{(12)}} c_{i,j} e_j$  for  $i \in I_{(15)}$ . Take an element  $\ell_3 = \ell(C, I_{2a_2+1}) \in O(U_{11}) \cap R$  with the matrix

$$C = \begin{pmatrix} I_{b_{12}} & \{c_{i,j}\} & 0 \\ 0 & I_{b_{15}} & 0 \\ 0 & 0 & I_{b_{13}+b_{12}} \end{pmatrix}.$$

Then we have

$$\ell_3^{-1} \ell_2^{-1} \ell_1 Y' \subset V_{(12)}^1 \oplus U_{(15)}^0$$

where  $U_{(15)}^0 = U_{(15)}^+ \oplus U_{(15)}^- = U_{(15)} \cap (U_+ + U_-)$ . Since  $\dim(Z \cap \ell_3^{-1} \ell_2^{-1} \ell_1 V_{11}) = b_{14}$  by definition, we can take an element  $\ell_4 = \ell(I_{a'_1}, B)$  with some  $B \in O_{2a_2+1}(\mathbb{F})$  such that

$$Z \cap \ell_4 \ell_3^{-1} \ell_2^{-1} \ell_1 V_{11} = V_{(14)} = \mathbb{F}e_{d+a_1+1} \oplus \cdots \oplus \mathbb{F}e_{d+a_1+b_{14}}.$$

We have only to consider  $V_{\#} = \ell_4 \ell_3^{-1} \ell_2^{-1} \ell_1 V_{11}$  and  $V'_{\#} = \ell_4 \ell_3^{-1} \ell_2^{-1} \ell_1 V'_{11} = (\bigoplus_{j=1}^{11} V_{(j)}) \oplus V_{\#}$  in the following.

By the above arguments we have

$$V_{\#} \supset V_{(12)}^1 \oplus (U_{(15)}^0 \cap V_{\#}) \oplus V_{(14)} \quad \text{and} \quad V_{\#}^{\perp} \supset V_{(12)}^1 \oplus (U_{(15)}^0 \cap V_{\#}^{\perp}) \oplus V_{(14)}$$

with  $\dim(U_{(15)}^0 \cap V_{\#}^{\perp}) = b_{15}$ . Let  $V_{\#}^{(15)}$  denote the orthogonal space of  $U_{(15)}^0 \cap V_{\#}^{\perp}$  in  $U_{(15)}^0$ . Then

$$V_{\#}^{(15)} = \begin{cases} U_{(15)}^0 \cap V_{\#}^{\perp} = U_{(15)}^0 \cap V_{\#} & \text{if } \varepsilon = 0, \\ (U_{(15)}^0 \cap V_{\#}) \oplus \mathbb{F}u_0 & \text{if } \varepsilon = 1 \end{cases}$$

where  $u_0$  is an element of  $V_{\#}^{(15)}$  such that  $(u_0, u_0) = -1$ .

Consider the projections  $\pi_{+-} : U_{11} \rightarrow U_{11} \cap (U_+ + U_-) = U_{11} \cap (U_{(+)} \oplus U_{(-)})$  and  $\pi_Z : U_{11} \rightarrow Z_0$  with respect to the direct sum decomposition

$$U_{11} = (U_{11} \cap (U_+ + U_-)) \oplus Z_0$$

where  $Z_0 = Z \cap U_{11} = \mathbb{F}e_{d+a_1+1} \oplus \cdots \oplus \mathbb{F}e_{d+a_1+b_{14}+1}$ . Since  $\pi_{+-}(V_{\#})$  is the orthogonal space of  $Y'(V_{\#}) = U_{11} \cap X'(V'_{\#})$  in  $U_{11} \cap (U_+ + U_-)$ , we have

$$\pi_{+-}(V_{\#}) = \left( \bigoplus_{i \in I_{(12)} \sqcup \kappa(I_{(12)}) \sqcup I_{(13)} \sqcup \overline{I_{(13)}}} \mathbb{F}e_i \right) \oplus \left( \bigoplus_{i \in I_{(12)}} \mathbb{F}(e_{\bar{i}} - e_{\overline{\kappa(i)}}) \right) \oplus V_{\#}^{(15)}.$$

Put  $u_i = e_i$  for  $i \in I_{(12)} \sqcup I_{(13)} \sqcup \overline{I_{(13)}}$  and  $u_{\bar{i}} = e_{\bar{i}} - e_{\overline{\kappa(i)}}$  for  $i \in I_{(12)}$ .

First suppose  $\varepsilon = 0$ . Then  $\{u_i \mid i \in I_{(12)} \sqcup I_{(13)} \sqcup \overline{I_{(13)}} \sqcup \overline{I_{(12)}}$  is a basis of a complementary subspace of  $Y(V_{\#}) = X(V'_{\#}) = V_{(12)}^1 \oplus V_{\#}^{(15)}$  in  $\pi_{+-}(V_{\#})$ . The space  $V_{\#}$  uniquely defines vectors  $v_i$  ( $i \in I_{(12)} \sqcup I_{(13)} \sqcup \overline{I_{(13)}} \sqcup \overline{I_{(12)}}$ ) contained in  $\mathbb{F}e_{d+a_1+b_{14}+1} \oplus \cdots \oplus \mathbb{F}e_{d+a_1+b_{14}+1}$  (the orthogonal complement of  $U_{(14)} = V_{(14)} \oplus \overline{V}_{(14)}$  in  $Z_0$ ) such that  $u_i + v_i \in V_{\#}$ . Since

$$(u_i, u_j) = \begin{cases} 1 & \text{if } j = \bar{i}, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$(v_i, v_j) = \begin{cases} -1 & \text{if } j = \bar{i}, \\ 0 & \text{otherwise.} \end{cases}$$

So we can take an element  $\ell_5 = \ell(I_{a'_1}, B)$  with some  $B \in O_{2a_2+1}(\mathbb{F})$  such that

$$(3.17) \quad \begin{aligned} & \ell_5 v_i = e_{\lambda(i)}, \quad \ell_5 v_{\bar{i}} = -e_{\overline{\lambda(i)}} \quad \text{for } i \in I_{(12)} \\ & \text{and that } \ell_5 v_i = e_{\eta_{13}(i)}, \quad \ell_5 v_{\bar{i}} = -e_{\overline{\eta_{13}(i)}} \quad \text{for } i \in I_{(13)}. \end{aligned}$$

Thus we have

$$\ell_5 V_{\#} = V_{(12)}^1 \oplus V_{(12)}^2 \oplus V_{(12)}^3 \oplus V_{(13)} \oplus V_{(14)} \oplus V_{\#}^{(15)} = V_{(12)} \oplus V_{(13)} \oplus V_{(14)} \oplus V_{\#}^{(15)}.$$

Next suppose  $\varepsilon = 1$ . Then we can take a basis  $u_i$  ( $i \in \{0\} \sqcup I_{(12)} \sqcup I_{(13)} \sqcup \overline{I_{(13)}} \sqcup \overline{I_{(12)}}$ ) of a complementary subspace of  $Y(V_{\#}) = V_{(12)}^1 \oplus (U_{(15)}^0 \cap V_{\#})$  in  $\pi_{+-}(V_{\#})$ . The space  $U_{11} \cap V_{\#}$  uniquely defines vectors  $v_i$  ( $i \in \{0\} \sqcup I_{(12)} \sqcup I_{(13)} \sqcup \overline{I_{(13)}} \sqcup \overline{I_{(12)}}$ ) contained in  $\mathbb{F}e_{d+a_1+b_{14}+1} \oplus \cdots \oplus \mathbb{F}e_{\overline{d+a_1+b_{14}+1}}$  such that  $u_i + v_i \in V_{\#}$ . Since

$$(u_i, u_j) = \begin{cases} -1 & \text{if } i = j = 0, \\ 1 & \text{if } j = \bar{i}, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$(v_i, v_j) = \begin{cases} 1 & \text{if } i = j = 0, \\ -1 & \text{if } j = \bar{i}, \\ 0 & \text{otherwise.} \end{cases}$$

So we can take an element  $\ell_5 = \ell(I_{a'_1}, B)$  with some  $B \in O_{2a_2+1}(\mathbb{F})$  satisfying (3.17) and  $\ell_5 v_0 = e_{n+1}$ . Thus we have

$$\ell_5 V_{\#} = V_{(12)} \oplus V_{(13)} \oplus V_{(14)} \oplus (U_{(15)}^0 \cap V_{\#}) \oplus \mathbb{F}(u_0 + e_{n+1}).$$

Finally we have only to consider  $(O(U_{(15)}) \cap R)$ -orbit of

$$U_{(15)} \cap \ell_5 V_{\#} = \begin{cases} U_{(15)}^0 \cap V_{\#} & \text{if } \varepsilon = 0, \\ (U_{(15)}^0 \cap V_{\#}) \oplus \mathbb{F}(u_0 + e_{n+1}) & \text{if } \varepsilon = 1. \end{cases}$$

But this problem is already solved in [M13]. There exists an  $\ell_6 \in O(U_{(15)}) \cap R \cong \text{GL}_{b_{15}}(\mathbb{F})$  such that  $\ell_6(U_{(15)} \cap \ell_5 V_{\#}) = V_{(15)}^{\varepsilon}$ . Thus we have

$$\ell_6 \ell_5 V_{\#} = V_{(12)} \oplus V_{(13)} \oplus V_{(14)} \oplus V_{(15)}^{\varepsilon}$$

and hence  $\ell_6 \ell_5 V'_{\#} = (\bigoplus_{j=1}^{14} V_{(j)}) \oplus V_{(15)}^{\varepsilon}$ , proving Theorem 3.15.  $\square$

**3.7. Construction of elements in  $R_V|_{U_+}$ .** Assume  $V = V(b_1, \dots, b_{15}, \varepsilon)$ . We will construct elements in  $R_V|_{U_+}$  where  $R_V = \{g \in R \mid gV = V\}$ .

**Lemma 3.18.** *For  $j = 1, \dots, 14$ , let  $A = \{a_{i,k}\}_{i,k \in I_{(j)}}$  be an invertible matrix with the inverse matrix  $A^{-1} = \{b_{i,k}\}$ . Then we can construct an element  $h = h_{(j)}(A)$  of  $R_V$  such that :*

(i) *If  $j \in J_1 = \{1, 2, 3, 4, 5, 6, 10, 11, 14\}$ , then*

$$he_k = \sum_{i \in I_{(j)}} a_{i,k} e_i, \quad he_{\bar{k}} = \sum_{i \in I_{(j)}} b_{k,i} e_{\bar{i}}$$

*for  $k \in I_{(j)}$  and  $he_{\ell} = e_{\ell}$  for  $\ell \in I - \widetilde{I_{(j)}}$ .*

(ii) *If  $j \in J_2 = \{7, 8, 9, 13\}$ , then*

$$he_k = \sum_{i \in I_{(j)}} a_{i,k} e_i, \quad he_{\eta_j(k)} = \sum_{i \in I_{(j)}} a_{i,k} e_{\eta_j(i)}, \quad he_{\bar{k}} = \sum_{i \in I_{(j)}} b_{k,i} e_{\bar{i}}, \quad he_{\overline{\eta_j(k)}} = \sum_{i \in I_{(j)}} b_{k,i} e_{\overline{\eta_j(i)}}$$

for  $k \in I_{(j)}$  and  $he_\ell = e_\ell$  for  $\ell \in I - \widetilde{I_{(j)}}$ .

(iii) If  $j = 12$ , then

$$\begin{aligned} he_k &= \sum_{i \in I_{(12)}} a_{i,k} e_i, & he_{\kappa(k)} &= \sum_{i \in I_{(12)}} a_{i,k} e_{\kappa(i)}, & he_{\lambda(k)} &= \sum_{i \in I_{(12)}} a_{i,k} e_{\lambda(i)}, \\ he_{\overline{k}} &= \sum_{i \in I_{(12)}} b_{k,i} e_{\overline{i}}, & he_{\overline{\kappa(k)}} &= \sum_{i \in I_{(12)}} b_{k,i} e_{\overline{\kappa(i)}}, & he_{\overline{\lambda(k)}} &= \sum_{i \in I_{(12)}} b_{k,i} e_{\overline{\lambda(i)}} \end{aligned}$$

for  $k \in I_{(12)}$  and  $he_\ell = e_\ell$  for  $\ell \in I - \widetilde{I_{(12)}}$ .

*Proof.* Clearly  $hU_+ = U_+$ ,  $hU_- = U_-$  and  $hV = V$ . So we have only to prove that  $h$  preserves the bilinear form  $(\ , \ )$ . Namely

$$(he_k, e_\ell) = (e_k, h^{-1}e_\ell)$$

for  $k, \ell \in I$ .

Suppose  $j \in J_1$ . Then the equality is nontrivial only when  $(k, \ell) \in (\overline{I_{(j)}} \times I_{(j)}) \sqcup (I_{(j)} \times \overline{I_{(j)}})$ . If  $k, \ell \in I_{(j)}$ , then

$$(he_{\overline{k}}, e_\ell) = \left( \sum_{i \in I_{(j)}} b_{k,i} e_{\overline{i}}, e_\ell \right) = b_{k,\ell} = (e_{\overline{k}}, \sum_{i \in I_{(j)}} b_{i,\ell} e_i) = (e_{\overline{k}}, h^{-1}e_\ell)$$

and

$$(he_k, e_{\overline{\ell}}) = \left( \sum_{i \in I_{(j)}} a_{i,k} e_i, e_{\overline{\ell}} \right) = a_{\ell,k} = (e_k, \sum_{i \in I_{(j)}} a_{\ell,i} e_{\overline{i}}) = (e_k, h^{-1}e_{\overline{\ell}}).$$

So the assertion is proved. We can also prove the assertion for  $j \in J_2 \sqcup \{12\}$  in the same way.  $\square$

The index set  $I_+ = \{i \in I \mid e_i \in U_+\}$  is decomposed as

$$I_+ = \bigsqcup_{j \in \mathcal{I}_+} I_{(j)}$$

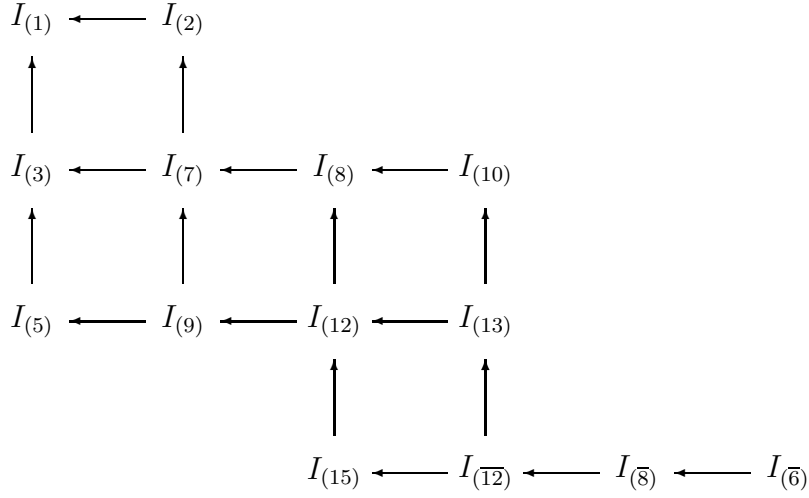
where  $\mathcal{I}_+ = \{1, 2, 3, 7, 8, 10, 5, 9, 12, 15, 13, \overline{12}, \overline{8}, \overline{6}\}$  and  $I_{(\overline{6})} = \overline{I_{(6)}}$ ,  $I_{(\overline{8})} = \overline{\eta_8(I_{(8)})}$  and  $I_{(\overline{12})} = \overline{\kappa(I_{(12)})}$ . Correspondingly  $U_+$  is decomposed as

$$U_+ = \bigoplus_{U \in \mathcal{U}} U$$

where

$$\begin{aligned} \mathcal{U} &= \{U_{(1)}^+, U_{(2)}^+, U_{(3)}^+, U_{(7)}^+, U_{(8)}^+, U_{(10)}^+, U_{(5)}^+, U_{(9)}^+, U_{(12)}^+, U_{(15)}^+, U_{(13)}^+, U_{(12)}^{+2}, U_{(8)}^{+2}, U_{(6)}^+\} \\ U_{(j)}^+ &= U_{(j)} \cap U_+ \ (j = 1, \dots, 15), \quad U_{(8)}^{+1} = \bigoplus_{i \in I_{(8)}} \mathbb{F}e_i, \quad U_{(8)}^{+2} = \bigoplus_{i \in I_{(\overline{8})}} \mathbb{F}e_i, \\ U_{(12)}^{+1} &= \bigoplus_{i \in I_{(12)}} \mathbb{F}e_i \quad \text{and} \quad U_{(12)}^{+2} = \bigoplus_{i \in I_{(\overline{12})}} \mathbb{F}e_i. \end{aligned}$$

Consider the following diagram of  $\{I_{(j)} \mid j \in \mathcal{I}_+\}$ .



We define a partial order  $I_{(j)} < I_{(j')}$  for  $j, j' \in \mathcal{I}_+$  if there exists a sequence  $j_0, j_1, \dots, j_k$  in  $\mathcal{I}_+$  such that

$$I_{(j)} = I_{(j_0)} \longleftarrow I_{(j_1)} \longleftarrow \dots \longleftarrow I_{(j_k)} = I_{(j')}.$$

For example,  $I_{(1)} < I_{(j)}$  for all  $j \in \mathcal{I}_+ - \{1\}$  and  $I_{(j)} < I_{(\overline{6})}$  for all  $j \in \mathcal{I}_+ - \{\overline{6}\}$ .

For  $i \in I_{(1)} \sqcup \dots \sqcup I_{(14)}$ , define  $I_{\langle j \rangle}, W_{\langle i \rangle}$  and  $\overline{W}_{\langle i \rangle}$  as in Remark 3.16. For  $i \in I_{(6)}$ , define

$$I_{\langle \overline{i} \rangle} = \{\overline{i}\}, \quad W_{\langle \overline{i} \rangle} = \mathbb{F}e_{\overline{i}} \quad \text{and} \quad \overline{W}_{\langle \overline{i} \rangle} = \mathbb{F}e_i.$$

For  $i \in I_{(8)}$ , define

$$I_{\langle \overline{\eta_8(i)} \rangle} = \{\overline{i}, \overline{\eta_8(i)}\}, \quad W_{\langle \overline{\eta_8(i)} \rangle} = \overline{W}_{\langle i \rangle} \quad \text{and} \quad \overline{W}_{\langle \overline{\eta_8(i)} \rangle} = W_{\langle i \rangle}.$$

For  $i \in I_{(12)}$ , define

$$I_{\langle \overline{\kappa(i)} \rangle} = \{\overline{i}, \overline{\kappa(i)}, \overline{\lambda(i)}\}, \quad W_{\langle \overline{\kappa(i)} \rangle} = \overline{W}_{\langle i \rangle} \quad \text{and} \quad \overline{W}_{\langle \overline{\kappa(i)} \rangle} = W_{\langle i \rangle}.$$

**Lemma 3.19.** *Suppose that  $I_{(j)} < I_{(j')}$  for  $j, j' \in \mathcal{I}_+ - \{15\}$ .*

(i) *For  $i \in I_{(j)}$  and  $k \in I_{(j')}$ , there exists a linear map  $\varphi = \varphi_{i,k} : W_{\langle k \rangle} \rightarrow W_{\langle i \rangle}$  satisfying the following three properties.*

(P1)  $\varphi(e_k) = e_i$ ,  $\varphi(W_{\langle k \rangle} \cap U_+^\perp) \subset U_+^\perp$ ,  $\varphi(W_{\langle k \rangle} \cap U_-) \subset U_-$ ,  $\varphi(W_{\langle k \rangle} \cap U_-^\perp) \subset U_-^\perp$  and  $\varphi(W_{\langle k \rangle} \cap V) \subset V$ .

(P2) *If  $i \in I_{(8)}$  and  $k \in I_{(12)}$ , then  $\varphi^*(e_{\overline{\eta_8(i)}}) = e_{\overline{\kappa(k)}}$ .*

(P3) *If  $i \in I_{(8)} \sqcup I_{(12)}$  and  $k \in I_{(\overline{8})} \sqcup I_{(\overline{12})}$ , then  $\varphi^*(\overline{W}_{\langle i \rangle} \cap U_+) = \{0\}$ .*

Here  $\varphi^* : \overline{W}_{\langle i \rangle} \rightarrow \overline{W}_{\langle k \rangle}$  is the adjoint map of  $\varphi = \varphi_{i,k}$  with respect to the bilinear form  $(\ , \ )$ .



$i \in I_{(j)}$	$e_i \in U_-$	$e_i \in U_-^\perp$	$e_i \in V$
$I_{(1)}$	Yes	Yes	Yes
$I_{(2)}$	Yes	Yes	No
$I_{(3)}$	No	Yes	Yes
$I_{(5)}$	No	No	Yes
$I_{(8)}$	No	Yes	No
$I_{(10)}$	No	Yes	No
$I_{(\overline{8})}$	No	No	No
$I_{(\overline{6})}$	No	No	Yes

 TABLE 2. Properties on  $e_i$  for  $i \in I_{(j)}$ 

(ii) If  $W_{\langle i \rangle} \neq \overline{W}_{\langle k \rangle}$ , then the map  $g = g_{i,k}(\mu) : \mathbb{F}^{2n+1} \rightarrow \mathbb{F}^{2n+1}$  ( $\mu \in \mathbb{F}$ ) defined by

$$ge_\ell = \begin{cases} e_\ell + \mu\varphi(e_\ell) & \text{if } \ell \in I_{\langle k \rangle}, \\ e_\ell - \mu\varphi^*(e_\ell) & \text{if } \ell \in \overline{I}_{\langle i \rangle}, \\ e_\ell & \text{if } \ell \in I - I_{\langle k \rangle} - \overline{I}_{\langle i \rangle} \end{cases}$$

is an element of  $R_V$ .

(iii) For  $i \in I_{(12)}$  and  $\mu \in \mathbb{F}$ , the map  $g = g_{i,\overline{\kappa(i)}}(\mu)$  defined by

$$ge_{\overline{\kappa(i)}} = e_{\overline{\kappa(i)}} + \mu e_i, \quad ge_{\overline{i}} = e_{\overline{i}} - \mu e_{\kappa(i)} \quad \text{and} \quad ge_\ell = e_\ell \text{ for } \ell \in I - \{\overline{i}, \overline{\kappa(i)}\}$$

is an element of  $R_V$ .

(iv) For  $i \in I_{(8)}$  and  $\mu \in \mathbb{F}$ , the map  $g = g_{i,\overline{\eta_8(i)}}(\mu)$  defined by

$$ge_{\overline{\eta_8(i)}} = e_{\overline{\eta_8(i)}} + \mu e_i, \quad ge_{\overline{i}} = e_{\overline{i}} - \mu e_{\eta_8(i)} \quad \text{and} \quad ge_\ell = e_\ell \text{ for } \ell \in I - \{\overline{i}, \overline{\eta_8(i)}\}$$

is an element of  $R_V$ .

*Proof.* (i) (A) First consider the case of  $I_{(j)} \longleftarrow I_{(j')}$ .

When  $\dim W_{\langle k \rangle} = 1$ , the pair  $(i, k)$  is contained in one of

$$I_{(1)} \times I_{(2)}, \quad I_{(1)} \times I_{(3)}, \quad I_{(3)} \times I_{(5)}, \quad I_{(8)} \times I_{(10)} \quad \text{or} \quad I_{(\overline{8})} \times I_{(\overline{6})}.$$

Then Table 2 shows that the maps

$$\varphi_{i,k} : W_{\langle k \rangle} \ni e_k \mapsto e_i \in W_{\langle i \rangle}$$

satisfy the property (P1).

Next consider the case where  $\dim W_{\langle k \rangle} = 2$  and  $\dim W_{\langle i \rangle} = 1$ . Then  $(i, k)$  is contained in  $I_{(j)} \times I_{(j')}$  with

$$(j, j') = (2, 7), (3, 7), (5, 9) \text{ or } (10, 13).$$

Define  $\varphi(e_{\eta_{j'}(k)})$  as follows. Then the maps  $\varphi = \varphi_{i,k}$  satisfy (P1).

$I_{(j)} \longleftarrow I_{(j')} \quad \varphi(e_{\eta_{j'}(k)})$	
$I_{(2)} \longleftarrow I_{(7)} \quad -e_i$	
$I_{(3)} \longleftarrow I_{(7)} \quad 0$	
$I_{(5)} \longleftarrow I_{(9)} \quad 0$	
$I_{(10)} \longleftarrow I_{(13)} \quad -e_i$	

Consider the case of  $\dim W_{<k>} = \dim W_{<i>} = 2$ . There are two cases

$$(i, k) \in I_{(7)} \times I_{(8)} \quad \text{and} \quad (i, k) \in I_{(7)} \times I_{(9)}.$$

In these cases, we put  $\varphi(e_{\eta_j(k)}) = e_{\eta_7(i)}$  for  $j = 8$  and  $9$ . Then the maps  $\varphi = \varphi_{i,k}$  satisfy (P1).

The remaining cases are when  $i$  or  $k$  is in  $I_{(12)} \sqcup I_{(\overline{12})}$ . The pair  $(i, k)$  is contained in one of

$$I_{(8)} \times I_{(12)}, \quad I_{(9)} \times I_{(12)}, \quad I_{(12)} \times I_{(13)}, \quad I_{(13)} \times I_{(\overline{12})} \quad \text{or} \quad I_{(\overline{12})} \times I_{(\overline{8})}.$$

If  $(i, k) \in I_{(8)} \times I_{(12)}$ , then we put

$$\varphi(e_{\kappa(k)}) = e_{\eta_8(i)} \quad \text{and} \quad \varphi(e_{\lambda(k)}) = -e_i.$$

Hence the property (P2) :  $\varphi^*(e_{\overline{\eta_8(i)}}) = e_{\overline{\kappa(k)}}$  is satisfied.

If  $(i, k) \in I_{(9)} \times I_{(12)}$ , then we put

$$\varphi(e_{\kappa(k)}) = e_{\eta_9(i)} \quad \text{and} \quad \varphi(e_{\lambda(k)}) = e_{\eta_9(i)}.$$

If  $(i, k) \in I_{(12)} \times I_{(13)}$ , then we put

$$\varphi(e_{\eta_{13}(k)}) = e_{\kappa(i)}.$$

For  $\overline{\kappa(k)} \in I_{(\overline{12})}$  and  $i \in I_{(13)}$ , the map  $\varphi = \varphi_{i, \overline{\kappa(k)}} : W_{<\overline{\kappa(k)}>} = \overline{W}_{<k>} \rightarrow W_{<i>}$  is defined by

$$\varphi(e_{\overline{\lambda(k)}}) = e_{\eta_{13}(i)} \quad \text{and} \quad \varphi(e_{\overline{k}}) = 0 \quad (\varphi(e_{\overline{\kappa(k)}}) = e_i \text{ by definition}).$$

For  $\overline{\eta_8(k)} \in I_{(\overline{8})}$  and  $\overline{\kappa(i)} \in I_{(\overline{12})}$ , the map  $\varphi = \varphi_{\overline{\kappa(i)}, \overline{\eta_8(k)}} : W_{<\overline{\eta_8(k)}>} = \overline{W}_{<k>} \rightarrow W_{<\overline{\kappa(i)}>} = \overline{W}_{<i>}$  is defined by

$$\varphi(e_{\overline{k}}) = e_{\overline{i}} - e_{\overline{\lambda(i)}} \quad (\varphi(e_{\overline{\eta_8(k)}}) = e_{\overline{\kappa(i)}} \text{ by definition}).$$

(Remark: The map  $\varphi_{\overline{\kappa(i)}, \overline{\eta_8(k)}} : W_{<\overline{\eta_8(k)}>} = \overline{W}_{<k>} \rightarrow W_{<\overline{\kappa(i)}>} = \overline{W}_{<i>}$  is the adjoint map of  $\varphi_{k,i} : W_{<i>} \rightarrow W_{<k>}$  for  $I_{(12)} \rightarrow I_{(8)}$  with respect to the bilinear form  $(\ , \ )$ .)

(B) Suppose we have constructed  $\varphi_{i,\ell} : W_{<\ell>} \rightarrow W_{<i>}$  and  $\varphi_{\ell,k} : W_{<k>} \rightarrow W_{<\ell>}$  for  $(i, \ell, k) \in I_{(j)} \times I_{(j'')} \times I_{(j')}$  with  $I_{(j)} < I_{(j'')} < I_{(j')}$ . Then the composition

$$\varphi = \varphi_{i,\ell} \circ \varphi_{\ell,k} : W_{<k>} \rightarrow W_{<i>}$$

also satisfies (P1). If the numbers  $b_1, \dots, b_{13}$  are all nonzero, then we can construct all  $\varphi$  in this way. If some of  $b_j$ 's are 0, then we have only to embed  $\mathbb{F}^{2n+1}$  into a larger space  $\mathbb{F}^{2n'+1}$  with nonzero  $b_j$ 's.

Remark: This construction of  $\varphi = \varphi_{i,\ell} \circ \varphi_{\ell,k}$  depends on the choice of intermediate  $j'' \in \mathcal{I}_+$ . For example, consider  $i \in I_{(1)}$  and  $k \in I_{(7)}$ . If  $\ell \in I_{(2)}$ , then

$$\varphi_{i,\ell} \circ \varphi_{\ell,k}(e_{\eta_7(k)}) = \varphi_{i,\ell}(-e_\ell) = -e_i.$$

But if  $\ell \in I_{(3)}$ , then

$$\varphi_{i,\ell} \circ \varphi_{\ell,k}(e_{\eta_7(k)}) = \varphi_{i,\ell}(0) = 0.$$

Suppose  $i \in I_{(8)} \sqcup I_{(12)}$  and  $k \in I_{(\overline{8})} \sqcup I_{(\overline{12})}$ . Then the map  $\varphi = \varphi_{i,k}$  is constructed as the composition of  $\varphi_{i,\ell} : W_{<\ell>} \rightarrow W_{<i>}$  and  $\varphi_{\ell,k} : W_{<k>} \rightarrow W_{<\ell>}$  with  $\ell \in I_{(13)}$ . Since  $\overline{W}_{<\ell>} \cap U_+ = \{0\}$  and since  $\varphi_{i,\ell}^*(\overline{W}_{<i>} \cap U_+) \subset \overline{W}_{<\ell>} \cap U_+$  by (P1), we have

$$\varphi^*(\overline{W}_{<i>} \cap U_+) = \varphi_{\ell,k}^* \varphi_{i,\ell}^*(\overline{W}_{<i>} \cap U_+) \subset \varphi_{\ell,k}^*(\overline{W}_{<\ell>} \cap U_+) = \{0\}.$$

So the map  $\varphi$  satisfies the property (P3).

(ii) It follows from  $\varphi(W_{<k>} \cap U_+^\perp) \subset U_+^\perp$ ,  $\varphi(W_{<k>} \cap U_-^\perp) \subset U_-^\perp$  and  $\varphi(W_{<k>} \cap V) \subset V$  that

$$(3.18) \quad \varphi^*(\overline{W}_{<i>} \cap U_+) \subset U_+, \quad \varphi^*(\overline{W}_{<i>} \cap U_-) \subset U_- \quad \text{and} \quad \varphi^*(\overline{W}_{<i>} \cap V) \subset V.$$

Put  $W_{<0>} = \bigoplus_{\ell \in I - I_{<k>} - \overline{I}_{<i>}} \mathbb{F}e_\ell$ . Then we have

$$\begin{aligned} U_+ &= (U_+ \cap W_{<k>}) \oplus (U_+ \cap \overline{W}_{<i>}) \oplus (U_+ \cap W_{<0>}), \\ U_- &= (U_- \cap W_{<k>}) \oplus (U_- \cap \overline{W}_{<i>}) \oplus (U_- \cap W_{<0>}) \\ \text{and } V &= (V \cap W_{<k>}) \oplus (V \cap \overline{W}_{<i>}) \oplus (V \cap W_{<0>}). \end{aligned}$$

Hence we have

$$gU_+ = U_+, \quad gU_- = U_- \quad \text{and} \quad gV = V$$

by (P1) and (3.18). So we have only to show that

$$(ge_\ell, e_{\ell'}) = (e_\ell, g^{-1}e_{\ell'})$$

for all  $\ell, \ell' \in I$ . This equality is trivial unless  $(\ell, \ell') \in (I_{<k>} \times \overline{I}_{<i>}) \sqcup (\overline{I}_{<i>} \times I_{<k>})$ . If  $(\ell, \ell') \in I_{<k>} \times \overline{I}_{<i>}$ , then

$$\begin{aligned} (ge_\ell, e_{\ell'}) &= (e_\ell + \mu\varphi(e_\ell), e_{\ell'}) = \mu(\varphi(e_\ell), e_{\ell'}) \\ &= \mu(e_\ell, \varphi^*(e_{\ell'})) = (e_\ell, e_{\ell'} + \mu\varphi^*(e_{\ell'})) = (e_\ell, g^{-1}e_{\ell'}). \end{aligned}$$

For  $(\ell, \ell') \in \overline{I}_{<i>} \times I_{<k>}$ , we also have the equality in the same way.

(iii) and (iv) are clear. □

For  $k \in I_{(15)}$ , define a vector  $f_k \in U_{(15)}^+$  by

$$f_k = \begin{cases} \delta(k)e_{\overline{\eta_{15}(k)}} & \text{if } k \neq c, \overline{\eta_{15}(c)}, \\ e_{\overline{\eta_{15}(c)}} + \frac{1}{2}\varepsilon e_c & \text{if } b_{15} \text{ is even and } k = c, \\ -e_{\overline{\eta_{15}(k)}} & \text{if } b_{15} \text{ is even and } k = \overline{\eta_{15}(c)}, \\ \frac{1}{2}e_c & \text{if } b_{15} \text{ is odd and } k = c \end{cases}$$

where

$$\delta(k) = \begin{cases} 1 & \text{if } k \in I_{(15)}^+, \\ -1 & \text{otherwise.} \end{cases}$$

For  $k \in I_{(15)}$ , define  $I_{<k>}$  by

$$I_{<k>} = \begin{cases} \{k, \eta_{15}(k), \bar{k}, \overline{\eta_{15}(k)}\} & \text{if } k \neq c, \overline{\eta_{15}(c)}, \\ \{c, \eta_{15}(c), \bar{c}, \overline{\eta_{15}(c)}, n+1\} & \text{if } b_{15} \text{ is even and } k = c, \overline{\eta_{15}(c)}, \\ \{c, \bar{c}, n+1\} & \text{if } b_{15} \text{ is odd and } k = c (= \overline{\eta_{15}(c)}) \end{cases}$$

and put  $W_{<k>} = \bigoplus_{\ell \in I_{<k>}} \mathbb{F}e_\ell$  (c.f. Remark 3.16).

**Lemma 3.20.** *Let  $i \in I_{(j)}$  with  $j = 1, 2, 3, 5, 7, 8, 9, 12$  and  $k \in I_{(15)}$ .*

(i) *There exists a linear map  $\varphi = \varphi_{i,k} : W_{<k>} \rightarrow W_{<i>}$  such that*

$$(3.19) \quad \begin{aligned} \varphi(e_k) &= e_i, \quad \varphi(W_{<k>} \cap U_+^\perp) \subset U_+^\perp, \quad \varphi(W_{<k>} \cap U_-) \subset U_-, \\ \varphi(W_{<k>} \cap U_-^\perp) &\subset U_-^\perp \quad \text{and that} \quad \varphi(W_{<k>} \cap V^\perp) \subset V^\perp. \end{aligned}$$

(Note that  $\varphi(W_{<k>} \cap V^\perp) \subset V^\perp$  implies  $\varphi(W_{<k>} \cap V^\perp) \subset W_{<i>} \cap V^\perp = W_{<i>} \cap V$  and hence  $\varphi(W_{<k>} \cap V) \subset V$ .)

(ii) *The map  $g = g'_{i,k}(\mu) : \mathbb{F}^{2n+1} \rightarrow \mathbb{F}^{2n+1}$  ( $\mu \in \mathbb{F}$ ) defined by*

$$ge_\ell = \begin{cases} e_\ell + \mu\varphi(e_\ell) & \text{if } \ell \in I_{<k>}, \\ e_\ell - \mu\varphi^*(e_\ell) - \frac{\mu^2}{2}\varphi\varphi^*(e_\ell) & \text{if } \ell \in \bar{I}_{<i>}, \\ e_\ell & \text{if } \ell \in I - I_{<k>} - \bar{I}_{<i>} \end{cases}$$

*is an element of  $R_V$ .*

(iii) *If  $i \in I_{(12)}$ , then  $g'_{i,k}(\mu)e_{\overline{\kappa(i)}} = e_{\overline{\kappa(i)}} - \mu f_k - \frac{\mu^2}{4}\varepsilon\delta_{k,c}e_i$ .*

(iv) *If  $i \in I_{(8)}$ , then  $g'_{i,k}(\mu)e_{\overline{\eta_8(i)}} = e_{\overline{\eta_8(i)}} - \mu f_k - \frac{\mu^2}{4}\varepsilon\delta_{k,c}e_i$ .*

*Proof.* (i) (A) First suppose that  $i \in I_{(12)}$ . If  $k \neq c, \overline{\eta_{15}(c)}$ , then  $\varphi = \varphi_{i,k} : W_{<k>} \rightarrow W_{<i>}$  is defined by

$$\varphi(e_k) = e_i, \quad \varphi(e_{\eta_{15}(k)}) = \delta(k)e_{\kappa(i)} \quad \text{and} \quad \varphi(e_{\bar{k}}) = \varphi(e_{\overline{\eta_{15}(k)}}) = 0.$$

If  $b_{15}$  is even and  $\varepsilon = 0$ , then  $\varphi = \varphi_{i,k} : W_{<c>} \rightarrow W_{<i>}$  for  $k = c, \overline{\eta_{15}(c)}$  are defined by

$$\varphi(e_k) = e_i, \quad \varphi(e_{\eta_{15}(k)}) = \delta(k)e_{\kappa(i)} \quad \text{and} \quad \varphi(e_{\bar{k}}) = \varphi(e_{\overline{\eta_{15}(k)}}) = \varphi(e_{n+1}) = 0.$$

Suppose  $b_{15}$  is even and  $\varepsilon = 1$ . Then  $\varphi = \varphi_{i,c} : W_{<c>} \rightarrow W_{<i>}$  is defined by

$$\begin{aligned} \varphi(e_c) &= e_i, \quad \varphi(e_{\eta_{15}(c)}) = e_{\kappa(i)}, \quad \varphi(e_{\bar{c}}) = \frac{1}{2}e_{\kappa(i)}, \\ \varphi(e_{\overline{\eta_{15}(c)}}) &= 0 \quad \text{and} \quad \varphi(e_{n+1}) = -e_{\lambda(i)}. \end{aligned}$$

(Since  $W_{<c>} \cap V^\perp = \mathbb{F}(e_{\bar{c}} - e_{\overline{\eta_{15}(c)}} + \frac{1}{2}e_c) \oplus \mathbb{F}(-e_c + e_{n+1})$ , we can verify the condition  $\varphi(W_{<c>} \cap V^\perp) \subset V^\perp$ .)  $\varphi = \varphi_{i,\overline{\eta_{15}(c)}} : W_{<c>} \rightarrow W_{<i>}$  is defined by

$$\varphi(e_{\overline{\eta_{15}(c)}}) = e_i, \quad \varphi(e_{\bar{c}}) = -e_{\kappa(i)} \quad \text{and} \quad \varphi(e_c) = \varphi(e_{\eta_{15}(c)}) = \varphi(e_{n+1}) = 0.$$

If  $b_{15}$  is odd, then  $\varphi = \varphi_{i,c} : W_{<c>} \rightarrow W_{<i>}$  is defined by

$$\varphi(e_c) = e_i, \quad \varphi(e_{\bar{c}}) = \frac{1}{2}e_{\kappa(i)} \quad \text{and} \quad \varphi(e_{n+1}) = -e_{\lambda(i)}.$$

(Since  $W_{<c>} \cap V^\perp = \mathbb{F}(e_{\bar{c}} + \frac{1}{2}e_c) \oplus \mathbb{F}(-e_c + e_{n+1})$ , we can verify the condition  $\varphi(W_{<c>} \cap V^\perp) \subset V^\perp$ .)

(B) When  $i \in I_{(j)}$  with  $j = 1, 2, 3, 5, 7, 8, 9$  and  $b_{12} \neq 0$ , we take the composition  $\varphi = \varphi_{i,\ell} \circ \varphi_{\ell,k}$  where  $\varphi_{i,\ell}$  is given in Lemma 3.19 and  $\varphi_{\ell,k}$  is given in (A) for  $\ell \in I_{(12)}$ . This construction is also valid for the case of  $b_{12} = 0$  as is explained in the proof of Lemma 3.19.

(ii) It follows from (3.19) that

$$\varphi^*(\overline{W}_{<i>} \cap U_+) \subset U_+, \quad \varphi^*(\overline{W}_{<i>} \cap U_-) \subset U_- \quad \text{and} \quad \varphi^*(\overline{W}_{<i>} \cap V) \subset V.$$

So we have  $gU_+ = U_+$ ,  $gU_- = U_-$  and  $gV = V$  as in the proof of Lemma 3.19. Since  $\det g = 1$ , we have only to show that

$$(ge_\ell, e_{\ell'}) = (e_\ell, g^{-1}e_{\ell'})$$

for all  $\ell, \ell' \in I$ . In view of the proof of Lemma 3.19, this equality is clear unless  $(\ell, \ell') \in I_{(j)} \times I_{(j)}$ . If  $(\ell, \ell') \in I_{(j)} \times I_{(j)}$ , then

$$(ge_\ell, e_{\ell'}) = -\frac{\mu^2}{2}(\varphi\varphi^*(e_\ell), e_{\ell'}) = -\frac{\mu^2}{2}(e_\ell, \varphi\varphi^*(e_{\ell'})) = (e_\ell, g^{-1}e_{\ell'})$$

since

$$g^{-1}e_{\ell'} = e_{\ell'} + \mu\varphi^*(e_{\ell'}) - \frac{\mu^2}{2}\varphi\varphi^*(e_{\ell'}).$$

(iii) It follows from the construction of  $\varphi$  in (i) that  $\varphi^*(e_{\overline{\kappa(i)}}) = f_k$ . Hence

$$\begin{aligned} ge_{\overline{\kappa(i)}} &= e_{\overline{\kappa(i)}} - \mu\varphi^*(e_{\overline{\kappa(i)}}) - \frac{\mu^2}{2}\varphi\varphi^*(e_{\overline{\kappa(i)}}) = e_{\overline{\kappa(i)}} - \mu f_k - \frac{\mu^2}{2}\varphi(f_k) \\ &= e_{\overline{\kappa(i)}} - \mu f_k - \frac{\mu^2}{4}\varepsilon\delta_{k,c}e_i. \end{aligned}$$

(iv) The proof is similar to (iii). □

**Lemma 3.21.** (i) Suppose that  $I_{(j)} < I_{(j')}$  for  $j, j' \in \mathcal{I}_+$  and that

$$(j, j') \notin \{(8, 12), (8, 15), (12, 15), (15, \overline{12}), (15, \overline{8}), (\overline{12}, \overline{8})\}.$$

Then for  $i \in I_{(j)}$ ,  $k \in I_{(j')}$  and  $\mu \in \mathbb{F}$ , there exists an element  $g = g_{i,k}(\mu) \in R_V$  such that

$$ge_k = e_k + \mu e_i \quad \text{and that} \quad ge_\ell = e_\ell \text{ for } \ell \in I_+ - \{k\}.$$

(ii) For  $i \in I_{(8)}$ ,  $k \in I_{(12)}$  and  $\mu \in \mathbb{F}$ , there exists an element  $g = g_{i,k}(\mu) \in R_V$  such that

$$ge_k = e_k + \mu e_i, \quad ge_{\overline{\eta_8(i)}} = e_{\overline{\eta_8(i)}} - \mu e_{\overline{\kappa(k)}}$$

and that  $ge_\ell = e_\ell$  for  $\ell \in I_+ - \{k, \overline{\eta_8(i)}\}$ .

(iii) For  $i \in I_{(12)}$ ,  $k \in I_{(15)}$  and  $\mu \in \mathbb{F}$ , there exists an element  $g = g_{i,k}(\mu) \in R_V$  such that

$$ge_k = e_k + \mu e_i, \quad ge_{\overline{\kappa(i)}} = e_{\overline{\kappa(i)}} - \mu f_k$$

and that  $ge_\ell = e_\ell$  for  $\ell \in I_+ - \{k, \overline{\kappa(i)}\}$ .

(iv) For  $i \in I_{(8)}$ ,  $k \in I_{(15)}$  and  $\mu \in \mathbb{F}$ , there exists an element  $g = g_{i,k}(\mu) \in R_V$  such that

$$ge_k = e_k + \mu e_i, \quad ge_{\overline{\eta_8(i)}} = e_{\overline{\eta_8(i)}} - \mu f_k$$

and that  $ge_\ell = e_\ell$  for  $\ell \in I_+ - \{k, \overline{\eta_8(i)}\}$ .

*Proof.* (i) and (ii). If  $j, j' \neq 15$ , then  $g = g_{i,k}(\mu)$  is constructed in Lemma 3.19. In particular, if  $i \in I_{(8)}$  and  $k \in I_{(12)}$ , then the property (P2) implies the assertion (ii).

Suppose  $j' = 15$  and  $j \neq 8, 12$ . Then  $g = g_{i,k}(\mu) = g'_{i,k}(\mu)$  is constructed in Lemma 3.20.

Finally consider the case of  $(j, j') = (15, \bar{6})$ . For  $i \in I_{(15)}$ ,  $k \in \bar{I}_{(6)}$  and  $\mu \in \mathbb{F}$ , we can define  $g = g_{i,k}(\mu) \in R_V$  by

$$ge_k = e_k + \mu e_i, \quad ge_{\bar{i}} = e_{\bar{i}} - \mu e_{\bar{k}}$$

and  $ge_\ell = e_\ell$  for  $\ell \in I - \{k, \bar{i}\}$ .

(iii) In Lemma 3.20, we constructed for  $i \in I_{(12)}$ ,  $k \in I_{(15)}$  and  $\mu \in \mathbb{F}$ ,  $g' = g'_{i,k}(\mu) \in R_V$  such that

$$g'e_k = e_k + \mu e_i \quad \text{and that} \quad g'e_{\overline{\kappa(i)}} = e_{\overline{\kappa(i)}} - \mu f_k - \frac{\mu^2}{4} \varepsilon \delta_{k,c} e_i.$$

So the element

$$g = g_{i,\overline{\kappa(i)}} \left( \frac{\mu^2}{4} \varepsilon \delta_{k,c} \right) g'$$

satisfies the desired property in view of Lemma 3.19 (iii).

The proof of (iv) is similar to (iii). □

**Lemma 3.22.** *Let  $k$  be an index in  $I_{(\bar{6})}$ . Then for any element  $u$  in  $\bigoplus_{i \in I - I_{(\bar{6})}} \mathbb{F}e_i$ , there exists a  $g \in R_V$  such that  $g(e_k + u) = e_k$ .*

*Proof.* For  $u = \sum_{i \in I_+ - I_{(\bar{6})}} \mu_i e_i$ , put  $g = \prod_{i \in I_+ - I_{(\bar{6})}} g_{i,k}(-\mu_i)$ . (The elements  $g_{i,k}(-\mu_i)$  are commutative for  $i \in I_+ - I_{(\bar{6})}$ .) Then we have  $g(e_k + u) = e_k$  by Lemma 3.18 (i). □

**Lemma 3.23.** *Let  $k$  be an index in  $I_{(8)}$ . Then for any element  $u$  in  $\bigoplus_{i \in I_+ - I_{(\bar{8})} - I_{(\bar{6})}} \mathbb{F}e_i$ , there exists a  $g \in R_V$  such that  $g(e_{\overline{\eta_8(k)}} + u) = e_{\overline{\eta_8(k)}}$  and that  $g$  acts trivially on  $U_{(6)}^+$ .*

*Proof.* Write  $u = u_1 + \sum_{i \in I_{(12)}} \mu_i e_{\overline{\kappa(i)}}$  with an element  $u_1 \in \bigoplus_{i \in I_+ - I_{(\bar{12})} - I_{(\bar{8})} - I_{(\bar{6})}} \mathbb{F}e_i$  and put  $g_1 = \prod_{i \in I_{(12)}} g_{k,i}(\mu_i)$ . Then we have

$$\begin{aligned} g_1(e_{\overline{\eta_8(k)}} + u) &= (e_{\overline{\eta_8(k)}} - \sum_{i \in I_{(12)}} \mu_i e_{\overline{\kappa(i)}}) + (g_1(u_1) + \sum_{i \in I_{(12)}} \mu_i e_{\overline{\kappa(i)}}) \\ &= e_{\overline{\eta_8(k)}} + g_1(u_1) \in e_{\overline{\eta_8(k)}} + \bigoplus_{i \in I_+ - I_{(\bar{12})} - I_{(\bar{8})} - I_{(\bar{6})}} \mathbb{F}e_i \end{aligned}$$

by Lemma 3.21 (ii).

Since  $\{f_i \mid i \in I_{(15)}\}$  is a basis of  $U_{(15)}^+$ , we can write  $g_1(u_1) = u_2 + \sum_{i \in I_{(15)}} \nu_i f_i$  with an element  $u_2 \in \bigoplus_{i \in I_+ - I_{(15)} - I_{(\overline{12})} - I_{(\overline{8})} - I_{(\overline{6})}} \mathbb{F}e_i$ . Put  $g_2 = \prod_{i \in I_{(15)}} g_{k,i}(-\nu_i)$ . Then we can write

$$g_2 e_{\overline{\eta_8(k)}} = e_{\overline{\eta_8(k)}} + \left( \sum_{i \in I_{(15)}} \nu_i f_i \right) + v$$

with some  $v \in U_{(8)}^{+1}$  by Lemma 3.21 (iv). So we have

$$\begin{aligned} g_2^{-1} g_1(e_{\overline{\eta_8(k)}} + u) &= g_2^{-1}(e_{\overline{\eta_8(k)}} + g_1(u_1)) = g_2^{-1}(e_{\overline{\eta_8(k)}} + \left( \sum_{i \in I_{(15)}} \nu_i f_i \right) + v + u_2 - v) \\ &= e_{\overline{\eta_8(k)}} + u_2 - v \in e_{\overline{\eta_8(k)}} + \bigoplus_{i \in I_+ - I_{(15)} - I_{(\overline{12})} - I_{(\overline{8})} - I_{(\overline{6})}} \mathbb{F}e_i. \end{aligned}$$

Write  $u_2 - v = \sum_{i \in I'} \rho_i e_i$  where  $I' = I_{(1)} \sqcup I_{(2)} \sqcup I_{(3)} \sqcup I_{(7)} \sqcup I_{(8)} \sqcup I_{(10)} \sqcup I_{(5)} \sqcup I_{(9)} \sqcup I_{(12)} \sqcup I_{(13)}$  and put  $g_3 = \prod_{i \in I'} g_{i, \overline{\eta_8(k)}}(-\rho_i)$ . Then we have

$$g_3 g_2^{-1} g_1(e_{\overline{\eta_8(k)}} + u) = g_3(e_{\overline{\eta_8(k)}} + u_2 - v) = e_{\overline{\eta_8(k)}}$$

by Lemma 3.21 (i). It follows from the definitions of  $g_1, g_2$  and  $g_3$  that  $g_3 g_2^{-1} g_1$  acts trivially on  $U_{(6)}^+$ .  $\square$

#### 4. FINITENESS OF $\mathcal{T}_{(\alpha_1, \alpha_2), (\beta), (n)}$

Put  $\alpha = \alpha_1 + \alpha_2$ . As in Section 3, we may fix  $\alpha$  and  $\beta$ -dimensional isotropic subspaces  $U_+$  and  $U_-$ , respectively. By Theorem 3.15, we may also fix a maximally isotropic subspace  $V = V(b_1, \dots, b_{15}, \varepsilon)$ . We have only to show that there are a finite number of  $R_V$ -orbits on the Grassmann variety of  $\alpha_1$ -dimensional subspaces of  $U_+$ .

For any subset  $J$  of  $I_+$ , let  $p_J$  denote the canonical projection of  $U_+$  onto  $\bigoplus_{i \in J} \mathbb{F}e_i$  defined by

$$p_J\left(\sum_{i \in I_+} a_i e_i\right) = \sum_{i \in J} a_i e_i.$$

**4.1. First reduction.** Let  $S$  be an  $\alpha_1$ -dimensional subspace of  $U_+$ . Put  $s_1 = \dim p_{I_{(\overline{6})}} S$ . Then there exists an  $h_1 \in R_V$  such that

$$p_{I_{(\overline{6})}} h_1 S = h_1 p_{I_{(\overline{6})}} S = U_{(6)}^{+, s_1} = \bigoplus_{i \in I_{(\overline{6}, s_1)}} \mathbb{F}e_i$$

where  $I_{(\overline{6}, s_1)} = \{d + a_1 - b_6 + 1, \dots, d + a_1 - b_6 + s_1\}$  by Lemma 3.18. Noting that  $\bigoplus_{i \in I_+ - I_{(\overline{6})}} \mathbb{F}e_i = U_+^{\perp(U_- \cap V)}$ , we can write

$$h_1 S = (h_1 S \cap U_+^{\perp(U_- \cap V)}) \oplus \left( \bigoplus_{i \in I_{(\overline{6}, s_1)}} \mathbb{F}v_i \right)$$

with vectors  $v_i \in e_i + U_+^{\perp(U_- \cap V)}$  for  $i \in I_{(\overline{6}, s_1)}$ . By Lemma 3.22, we can take an element  $g_1 \in R_V$  such that

$$g_1 v_i = e_i \quad \text{for all } i \in I_{(\overline{6}, s_1)}$$

and that  $g_1$  acts trivially on  $U_+^{\perp(U_- \cap V)}$ . So we have

$$g_1 h_1 S = S_1 \oplus U_{(6)}^{+,s_1}$$

where  $S_1 = g_1 h_1 S \cap U_+^{\perp(U_- \cap V)} = S \cap U_+^{\perp(U_- \cap V)}$ .

**4.2. Second reduction.** Put  $s_2 = \dim p_{I_{(8)}} S_1$ . Then there exists an  $h_2 \in R_V$  such that

$$p_{I_{(8)}} h_2 S_1 = h_2 p_{I_{(8)}} S_1 = U_{(8)}^{+,2,s_2} = \bigoplus_{i \in I_{(8,2)}} \mathbb{F} \overline{e_{\eta_8(i)}}$$

where  $I_{(8,2)} = \{a_0 + b_3 + b_7 + b_8 - s_2 + 1, \dots, a_0 + b_3 + b_7 + b_8\}$  by Lemma 3.18. Since  $\bigoplus_{i \in I_+ - I_{(8)} - I_{(6)}} \mathbb{F} e_i = U_+^{\perp((U_- + W_+) \cap V)}$ , we can write

$$h_2 S_1 = (h_2 S_1 \cap U_+^{\perp((U_- + W_+) \cap V)}) \oplus \left( \bigoplus_{i \in I_{(8,2)}} \mathbb{F} v_i \right)$$

with vectors  $v_i \in e_{\eta_8(i)} + U_+^{\perp((U_- + W_+) \cap V)}$  for  $i \in I_{(8,2)}$ . By Lemma 3.23, we can take an element  $g_2 \in R_V$  such that

$$g_2 v_i = e_{\eta_8(i)} \quad \text{for all } i \in I_{(8,2)}.$$

(Note that  $g_2$  does not act trivially on  $U_+^{\perp((U_- + W_+) \cap V)}$  in general.) So we have

$$g_2 h_2 S_1 = S_2 \oplus U_{(8)}^{+,2,s_2}$$

where  $S_2 = g_2 h_2 S_1 \cap U_+^{\perp((U_- + W_+) \cap V)} = g_2 S_1 \cap U_+^{\perp((U_- + W_+) \cap V)}$ . Since  $g_2$  and  $h_2$  acts trivially on  $U_{(6)}^+$ , we have

$$g_2 h_2 g_1 h_1 S = S_2 \oplus U_{(8)}^{+,2,s_2} \oplus U_{(6)}^{+,s_1}.$$

**4.3. Third reduction.** The index set  $I_{(8)}$  is decomposed as  $I_{(8)} = I_{(8,1)} \sqcup I_{(8,2)}$  where

$$I_{(8,1)} = \{a_0 + b_3 + b_7 + 1, \dots, a_0 + b_3 + b_7 + b_8 - s_2\}.$$

Let  $U_{(8)}^{+1} = U_{(8,1)}^{+1} \oplus U_{(8,2)}^{+1}$  be the corresponding direct sum decomposition of  $U_{(8)}^{+1}$  into

$$U_{(8,1)}^{+1} = \bigoplus_{i \in I_{(8,1)}} \mathbb{F} e_i \quad \text{and} \quad U_{(8,2)}^{+1} = \bigoplus_{i \in I_{(8,2)}} \mathbb{F} e_i.$$

Then we have easily the following lemma.

**Lemma 4.1.** *Let  $h = h_8(A)$  be the element of  $R_V$  defined in Lemma 3.18 for  $A \in \text{GL}(U_{(8)}^{+1})$ . Then*

$$h U_{(8)}^{+,2,s_2} = U_{(8)}^{+,2,s_2} \iff A U_{(8,1)}^{+1} = U_{(8,1)}^{+1}.$$



Write  $p = p_{I_{(8,2)}}$ ,  $q = p_{I_{(10)}}$  and  $r = p_{I_{(12)}}$ . Put  $U_0 = U_{(1)}^+ \oplus U_{(2)}^+ \oplus U_{(3)}^+ \oplus U_{(7)}^+ \oplus U_{(5)}^+ \oplus U_{(9)}^+ \oplus U_{(8)}^{+1}$ . Put  $s_3 = \dim p(S_2 \cap U_0)$  and  $s_4 = \dim p(S_2 \cap (U_0 \oplus U_{(12)}^{+1})) - s_3$ . Define a decomposition  $I_{(8,2)} = I_{(8,2,1)} \sqcup I_{(8,2,2)} \sqcup I_{(8,2,3)}$  where

$$I_{(8,2,1)} = \{t_0 + 1, \dots, t_0 + s_3\}, \quad I_{(8,2,2)} = \{t_0 + s_3 + 1, \dots, t_0 + s_3 + s_4\}$$

and  $I_{(8,2,3)} = \{t_0 + s_3 + s_4 + 1, \dots, t_0 + s_2\}$

( $t_0 = a_0 + b_3 + b_7 + b_8 - s_2$ ). Let  $U_{(8,2)}^{+1} = U_{(8,2,1)}^{+1} \oplus U_{(8,2,2)}^{+1} \oplus U_{(8,2,3)}^{+1}$  denote the corresponding direct sum decomposition. Then there exists an  $h_3 \in R_V$  such that

$$h_3 p(S_2 \cap U_0) = U_{(8,2,1)}^{+1}, \quad h_3 p(S_2 \cap (U_0 \oplus U_{(12)}^{+1})) = U_{(8,2,1)}^{+1} \oplus U_{(8,2,2)}^{+1}, \quad h_3 p = p h_3$$

and that  $h_3 e_i = e_i$  for all  $i \in I_+ - I_{(8,2)} - \overline{\eta_8(I_{(8,2)})}$  by Lemma 3.18.

Put  $s_5 = \dim q(S_2 \cap (U_0 \oplus U_{(10)}^+))$  and  $s_6 = \dim q(S_2 \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1})) - s_5$ . Define a decomposition  $I_{(10)} = I_{(10,1)} \sqcup I_{(10,2)} \sqcup I_{(10,3)}$  where

$$I_{(10,1)} = \{t_0 + s_2 + 1, \dots, t_0 + s_2 + s_5\},$$

$$I_{(10,2)} = \{t_0 + s_2 + s_5 + 1, \dots, t_0 + s_2 + s_5 + s_6\}$$

and  $I_{(10,3)} = \{t_0 + s_2 + s_5 + s_6 + 1, \dots, t_0 + s_2 + b_{10}\}.$

Let  $U_{(10)}^+ = U_{(10,1)}^+ \oplus U_{(10,2)}^+ \oplus U_{(10,3)}^+$  denote the corresponding direct sum decomposition. Then there exists an  $h_4 \in R_V$  such that

$$h_4 q(S_2 \cap (U_0 \oplus U_{(10)}^+)) = U_{(10,1)}^+, \quad h_4 q(S_2 \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1})) = U_{(10,1)}^+ \oplus U_{(10,2)}^+,$$

$h_4 q = q h_4$  and that  $h_4 e_i = e_i$  for all  $i \in I_+ - I_{(10)}$  by Lemma 3.18. Put  $S'_2 = h_4 h_3 S_2$ . Then we have

$$p(S'_2 \cap (U_0 \oplus U_{(12)}^{+1})) = U_{(8,2,1)}^{+1} \oplus U_{(8,2,2)}^{+1}$$

and  $q(S'_2 \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1})) = U_{(10,1)}^+ \oplus U_{(10,2)}^+.$

We can write

$$S'_2 \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1}) = (S'_2 \cap (U_0 \oplus U_{(12)}^{+1})) \oplus \bigoplus_{i \in I_{(10,1)} \sqcup I_{(10,2)}} \mathbb{F} v_i$$

with vectors  $v_i \in e_i + U_0 + U_{(12)}^{+1}$ . Define a subspace

$$U'_0 = U_{(1)}^+ \oplus U_{(2)}^+ \oplus U_{(3)}^+ \oplus U_{(7)}^+ \oplus U_{(5)}^+ \oplus U_{(9)}^+ \oplus U_{(8,1)}^{+1}$$

of  $U_0$ . Then  $U_0 = U'_0 \oplus U_{(8,2)}^{+1}$ . By Lemma 3.21 (i), we can take an element  $g_3 \in R_V$  such that

$$g_3 v_i \in e_i + U'_0 + U_{(12)}^{+1}$$

for  $i \in I_{(10,1)} \sqcup I_{(10,2)}$  and that  $g_3 e_i = e_i$  for  $i \in I_+ - (I_{(10,1)} \sqcup I_{(10,2)})$ . Put  $S''_2 = g_3 S'_2$ . Then

$$p_{I_{(8,2)} \sqcup I_{(10)}}(S''_2 \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1})) = U_{(8,2,1)}^{+1} \oplus U_{(8,2,2)}^{+1} \oplus U_{(10,1)}^+ \oplus U_{(10,2)}^+.$$

Put  $S_3 = S''_2 \cap (U_0 \oplus U_{(10)}^+)$ . Since  $p_{I_{(8,2)} \sqcup I_{(10)}} S_3 = U_{(8,2,1)}^{+1} \oplus U_{(10,1)}^+$ , we can take a complementary subspace  $S_4$  of  $S_3$  in  $S''_2 \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1})$  such that

$$(4.1) \quad p_{I_{(8,2)} \sqcup I_{(10)}} S_4 = U_{(8,2,2)}^{+1} \oplus U_{(10,2)}^+.$$

In the same way, we can take a complementary subspace  $S_5$  of  $S_3 \oplus S_4 = S_2'' \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1})$  in  $S_2'' \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1} \oplus U_{(15)}^+)$  and a complementary subspace  $S_6$  of  $S_3 \oplus S_4 \oplus S_5 = S_2'' \cap (U_0 \oplus U_{(10)}^+ \oplus U_{(12)}^{+1} \oplus U_{(15)}^+)$  in  $S_2''$  such that

$$p_{I_{(8,2)} \sqcup I_{(10)}}(S_5 \oplus S_6) \subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^+.$$

Hence  $S_2''$  is decomposed as

$$S_2'' = S_3 \oplus S_4 \oplus S_5 \oplus S_6.$$

**4.4. Finiteness of orbits for  $S_3$ -part.** Put  $\gamma = \dim S_3$ . We can show that there are a finite number of  $R_V$ -orbits on the Grassmann variety consisting of  $\gamma$ -dimensional subspaces of  $U_{(S_3)} = U_0' \oplus U_{(8,2,1)}^{+1} \oplus U_{(10,1)}^+$  as follows. The space  $U_{(S_3)}$  can be decomposed as

$$U_{(S_3)} = U_{(S_3,1)} \oplus U_{(S_3,2)}$$

where

$$U_{(S_3,1)} = U_{(1)}^+ \oplus U_{(2)}^+ \oplus U_{(7)}^+ \oplus U_{(8,1)}^{+1} \oplus U_{(8,2,1)}^{+1} \oplus U_{(10,1)}^+ \quad \text{and} \quad U_{(S_3,2)} = U_{(3)}^+ \oplus U_{(5)}^+ \oplus U_{(9)}^+.$$

By Lemma 3.18 and Lemma 3.21 (i),  $R_V|_{U_+}$  contains parabolic subgroups of  $\mathrm{GL}(U_{(S_3,1)})$  and  $\mathrm{GL}(U_{(S_3,2)})$  stabilizing the flags

$$\begin{aligned} U_{(1)}^+ &\subset U_{(1)}^+ \oplus U_{(2)}^+ \subset U_{(1)}^+ \oplus U_{(2)}^+ \oplus U_{(7)}^+ \subset U_{(1)}^+ \oplus U_{(2)}^+ \oplus U_{(7)}^+ \oplus U_{(8,1)}^{+1} \\ &\subset U_{(1)}^+ \oplus U_{(2)}^+ \oplus U_{(7)}^+ \oplus U_{(8,1)}^{+1} \oplus U_{(8,2,1)}^{+1} \end{aligned}$$

$$\text{and } U_{(3)}^+ \subset U_{(3)}^+ \oplus U_{(5)}^+,$$

respectively. (The first parabolic subgroup stabilizes  $U_{(8)}^{+2,s_2}$  by Lemma 4.1.) So it contains the product  $B_1 \times B_2$  of Borel subgroups. By Proposition 6.3 in the appendix, there are a finite number of  $B_1 \times B_2$ -orbits on the Grassmann variety. (Remark: We may also directly get explicit but complicated representatives.)

**4.5. A standard form of  $S_4$ .** The projection  $r = p_{I_{(12)}}$  is injective on  $S_4$ . So we have a bijection

$$r : \tilde{S}_4 \xrightarrow{\sim} r(S_4)$$

where  $\tilde{S}_4 = p_{I_{(8,2)} \sqcup I_{(10)} \sqcup I_{(12)}}(S_4)$ . By (4.1), we have a surjection

$$f = p_{I_{(8,2)} \sqcup I_{(10)}} \circ (r|_{\tilde{S}_4})^{-1} : r(S_4) \rightarrow U_{(8,2,2)}^{+1} \oplus U_{(10,2)}^+$$

with the kernel  $r(S_4 \cap (U_0' \oplus U_{(12)}^{+1}))$ . Put  $s_7 = \dim r(S_4 \cap (U_0' \oplus U_{(12)}^{+1}))$ . Then  $\dim r(S_4) = s_4 + s_6 + s_7$ . We can take a basis  $v_1, \dots, v_{s_4+s_6+s_7}$  of  $r(S_4)$  such that

$$r(S_4 \cap (U_0' \oplus U_{(12)}^{+1})) = \mathbb{F}v_1 \oplus \dots \oplus \mathbb{F}v_{s_7},$$

$$f(v_{s_7+i}) = e_{t_0+s_3+i} \text{ for } i = 1, \dots, s_4$$

$$\text{and that } f(v_{s_7+s_4+i}) = e_{t_0+s_2+s_5+i} \text{ for } i = 1, \dots, s_6.$$

Take an element  $A \in \mathrm{GL}(U_{(12)}^{+1})$  such that

$$Av_i = e_{d+b_5+b_9+i} \text{ for } i = 1, \dots, s_4 + s_6 + s_7$$

and put  $h_5 = h_{(12)}(A) \in R_V$ . Then

$$p_{I_{(8,2)} \sqcup I_{(10)} \sqcup I_{(12)}}(h_5 S_4) = h_5 \tilde{S}_4 = U(s_7, s_4, s_6)$$

where

$$\begin{aligned} U(s_7, s_4, s_6) = & \left( \bigoplus_{i=1}^{s_7} \mathbb{F} e_{d+b_5+b_9+i} \right) \oplus \left( \bigoplus_{i=1}^{s_4} \mathbb{F} (e_{d+b_5+b_9+s_7+i} + e_{t_0+s_3+i}) \right) \\ & \oplus \left( \bigoplus_{i=1}^{s_6} \mathbb{F} (e_{d+b_5+b_9+s_7+s_4+i} + e_{t_0+s_2+s_5+i}) \right). \end{aligned}$$

Define a decomposition  $I_{(12)} = I_{(12,1)} \sqcup I_{(12,2)} \sqcup I_{(12,3)} \sqcup I_{(12,4)}$  of  $I_{(12)}$  into subspaces

$$I_{(12,1)} = \{d + b_5 + b_9 + 1, \dots, d + b_5 + b_9 + s_7\},$$

$$I_{(12,2)} = \{d + b_5 + b_9 + s_7 + 1, \dots, d + b_5 + b_9 + s_7 + s_4\},$$

$$I_{(12,3)} = \{d + b_5 + b_9 + s_7 + s_4 + 1, \dots, d + b_5 + b_9 + s_7 + s_4 + s_6\}$$

$$\text{and } I_{(12,4)} = \{d + b_5 + b_9 + s_7 + s_4 + s_6 + 1, \dots, d + b_5 + b_9 + b_{12}\}.$$

Define bijections  $\sigma : I_{(12,2)} \rightarrow I_{(8,2,2)}$  and  $\tau : I_{(12,3)} \rightarrow I_{(10,2)}$  by

$$\sigma(d + b_5 + b_9 + s_7 + i) = t_0 + s_3 + i \quad \text{and} \quad \tau(d + b_5 + b_9 + s_7 + s_4 + i) = t_0 + s_2 + s_5 + i,$$

respectively. Then we can write

$$U(s_7, s_4, s_6) = \left( \bigoplus_{i \in I_{(12,1)}} \mathbb{F} e_i \right) \oplus \left( \bigoplus_{i \in I_{(12,2)}} \mathbb{F} (e_i + e_{\sigma(i)}) \right) \oplus \left( \bigoplus_{i \in I_{(12,3)}} \mathbb{F} (e_i + e_{\tau(i)}) \right).$$

**Lemma 4.2.** *For  $i \in I_{(12)}$  and  $u \in U'_0$ , there exists a  $g \in R_V$  such that  $g(e_i + u) = e_i$  and that  $g$  acts trivially on  $U_{(S_3)} \oplus U_{(8)}^{+2, s_2} \oplus U_{(6)}^{+, s_1}$ .*

*Proof.* Write  $u = u_1 + u_2$  with  $u_1 \in U_{(1)}^+ \oplus U_{(2)}^+ \oplus U_{(3)}^+ \oplus U_{(7)}^+ \oplus U_{(5)}^+ \oplus U_{(9)}^+$  and  $u_2 \in U_{(8,1)}^{+1}$ . By Lemma 3.21 (i), we can take an element  $g_1 \in R_V$  such that  $g_1(e_i + u_1) = e_i$  and that  $g_1(e_k) = e_k$  for  $k \in I_+ - \{i\}$ .

Write  $u_2 = \sum_{k \in I_{(8,1)}} \mu_k e_k$  and put  $g_2 = \prod_{k \in I_{(8,1)}} g_{i,k}(-\mu_k)$ . Then by Lemma 3.21 (ii), we have  $g_2(e_i + u_2) = e_i$  and  $g_2$  acts trivially on  $U_{(8)}^{+2, s_2}$  because  $\overline{\eta_8(i)} \notin \overline{\eta_8(I_{(8,2)})}$ . The element  $g = g_2 g_1$  satisfies the desired property.  $\square$

We can write

$$h_5 S_4 = \left( \bigoplus_{i \in I_{(12,1)}} \mathbb{F} (e_i + u_i) \right) \oplus \left( \bigoplus_{i \in I_{(12,2)}} \mathbb{F} (e_i + e_{\sigma(i)} + u_i) \right) \oplus \left( \bigoplus_{i \in I_{(12,3)}} \mathbb{F} (e_i + e_{\tau(i)} + u_i) \right).$$

with some  $u_i \in U'_0$  for  $i \in I_{(12,1)} \sqcup I_{(12,2)} \sqcup I_{(12,3)}$ . By Lemma 4.2, there exist  $g^{(i)} \in R_V$  for  $i \in I_{(12,1)} \sqcup I_{(12,2)} \sqcup I_{(12,3)}$  such that

$$g^{(i)}(e_i + u_i) = e_i$$

and that  $g^{(i)}$  acts trivially on  $U_{(S_3)} \oplus U_{(8)}^{+2, s_2} \oplus U_{(6)}^{+, s_1}$ . Put  $g_4 = \prod_{i \in I_{(12,1)} \sqcup I_{(12,2)} \sqcup I_{(12,3)}} g^{(i)}$ . Then

$$g_4 h_5 S_4 = U(s_7, s_4, s_6).$$

**4.6. Normalization of  $g_4h_5(S_5 \oplus S_6)$ .** Since  $g_4h_5S_2'' = S_3 \oplus g_4h_5S_4 \oplus g_4h_5(S_5 \oplus S_6) = S_3 \oplus U(s_7, s_4, s_6) \oplus g_4h_5(S_5 \oplus S_6)$ , we have only to consider the orbit of  $g_4h_5(S_5 \oplus S_6)$  by the action of the subgroup

$$\begin{aligned} R'_V &= \{g \in R_V \mid g(S_3 \oplus U(s_7, s_4, s_6) \oplus U_{(8)}^{+2, s_2} \oplus U_{(6)}^{+, s_1}) \\ &= S_3 \oplus U(s_7, s_4, s_6) \oplus U_{(8)}^{+2, s_2} \oplus U_{(6)}^{+, s_1}\} \end{aligned}$$

of  $R_V$ .

Since  $p_{I_{(15)}}$  is injective on  $g_4h_5S_5$ , there exist linear maps

$$f_1 : p_{I_{(15)}}(g_4h_5S_5) \rightarrow U_0' \oplus U_{(12)}^{+1} \quad \text{and} \quad f_2 : p_{I_{(15)}}(g_4h_5S_5) \rightarrow U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^+$$

such that

$$g_4h_5S_5 = \{u + f_1(u) + f_2(u) \mid u \in p_{I_{(15)}}(g_4h_5S_5)\}.$$

Extend  $f_1$  and  $f_2$  linearly on  $U_{(15)}^+$ . Then by Lemma 3.21 (i), (iii) and (iv), there exists an element  $g_5 \in R_V$  such that

$$g_5(u + f_1(u)) = u \quad \text{for } u \in U_{(15)}^+.$$

The element  $g_5$  is a product of elements of the form  $g_{i,k}(\mu)$  with  $k \in I_{(15)}$ ,  $i \in I_{(1)} \sqcup I_{(2)} \sqcup I_{(3)} \sqcup I_{(7)} \sqcup I_{(5)} \sqcup I_{(9)} \sqcup I_{(8,1)} \sqcup I_{(12)}$  and  $\mu \in \mathbb{F}$ . When  $i \in I_{(8,1)}$ ,  $e_{\overline{\eta_8(i)}}$  is not contained in  $U_{(8)}^{+2, s_2}$  and hence  $g_{i,k}(\mu)$  acts trivially on  $U_{(8)}^{+2, s_2}$ . Similarly we see that each  $g_{i,k}(\mu)$  acts trivially on  $S_3 \oplus U(s_7, s_4, s_6) \oplus U_{(8)}^{+2, s_2} \oplus U_{(6)}^{+, s_1}$ . Hence  $g_5 \in R'_V$ . (Note that  $g_{i,k}(\mu)$  for  $k \in I_{(12)}$  does not stabilize  $g_4h_5S_6$  in general.) Thus we have  $g_5 \in R'_V$  such that

$$g_5g_4h_5S_5 \subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^+ \oplus U_{(15)}^+.$$

Since  $p_{I_{(13)} \sqcup I_{(12)}}$  is injective on  $g_5g_4h_5S_5$ , there exist linear maps

$$\begin{aligned} f_3 : p_{I_{(13)} \sqcup I_{(12)}}(g_5g_4h_5S_5) &\rightarrow U_0' \oplus U_{(12)}^{+1} \\ \text{and } f_4 : p_{I_{(13)} \sqcup I_{(12)}}(g_5g_4h_5S_5) &\rightarrow U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^+ \oplus U_{(15)}^+ \end{aligned}$$

such that

$$g_5g_4h_5S_5 = \{u + f_3(u) + f_4(u) \mid u \in p_{I_{(13)} \sqcup I_{(12)}}(g_5g_4h_5S_5)\}.$$

Extend  $f_3$  and  $f_4$  linearly on  $U_{(13)}^+ \oplus U_{(12)}^{+2}$ . Then by Lemma 3.21 (i), there exists an element  $g_6 \in R_V$  such that

$$g_6(u + f_3(u)) = u \quad \text{for } u \in U_{(13)}^+ \oplus U_{(12)}^{+2}.$$

Since  $g_6e_i = e_i$  for  $i \in I_+ - I_{(13)} - I_{(12)}$ , we have  $g_6 \in R'_V$  and  $g_6$  acts trivially on  $g_5g_4h_5S_5$ . Thus we have

$$g_6g_5g_4h_5S_6 \subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^+ \oplus U_{(13)}^+ \oplus U_{(12)}^{+2} \oplus U_{(15)}^+$$

and

$$g_6g_5g_4h_5(S_5 \oplus S_6) \subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^+ \oplus U_{(13)}^+ \oplus U_{(12)}^{+2} \oplus U_{(15)}^+.$$

**4.7. Construction of a subgroup of  $R'_V$ .** Put  $U_{(12,j)}^{+1} = \bigoplus_{i \in I_{(12,j)}} \mathbb{F}e_i$  for  $j = 1, 2, 3, 4$ . Define linear isomorphisms  $\sigma : U_{(12,2)}^{+1} \xrightarrow{\sim} U_{(8,2,2)}^{+1}$  and  $\tau : U_{(12,3)}^{+1} \xrightarrow{\sim} U_{(10,2)}^{+1}$  by

$$\sigma(e_i) = e_{\sigma(i)} \quad \text{and} \quad \tau(e_i) = e_{\tau(i)},$$

respectively. For  $A_j \in \text{GL}(U_{(12,j)}^{+1})$ , define  $\widetilde{A}_j \in \text{GL}(U_{(12)}^{+1})$  by

$$\widetilde{A}_j v = \begin{cases} A_j v & \text{for } v \in U_{(12,j)}^{+1}, \\ v & \text{for } v \in U_{(12,k)}^{+1} \text{ with } k \neq j. \end{cases}$$

We also define  $\widetilde{B} \in \text{GL}(U_{(8)}^{+1})$  for  $B \in \text{GL}(U_{(8,2,2)}^{+1})$  and  $\widetilde{C} \in \text{GL}(U_{(10)}^{+1})$  for  $C \in \text{GL}(U_{(10,2)}^{+1})$  in the same way. Then the following lemma is clear from the definition of  $U(s_7, s_4, s_6)$ .

**Lemma 4.3.** (i) For  $A_1 \in \text{GL}(U_{(12,1)}^{+1})$  and  $A_4 \in \text{GL}(U_{(12,4)}^{+1})$ ,  $\widetilde{h}(A_1) = h_{(12)}(\widetilde{A}_1)$  and  $\widetilde{h}(A_4) = h_{(12)}(\widetilde{A}_4)$  are elements of  $R'_V$ .

(ii) For  $A_2 \in \text{GL}(U_{(12,2)}^{+1})$ ,  $\widetilde{h}(A_2) = h_{(8)}(\widetilde{\sigma A_2 \sigma^{-1}}) h_{(12)}(\widetilde{A}_2)$  is an element of  $R'_V$ .

(iii) For  $A_3 \in \text{GL}(U_{(12,3)}^{+1})$ ,  $\widetilde{h}(A_3) = h_{(10)}(\widetilde{\tau A_3 \tau^{-1}}) h_{(12)}(\widetilde{A}_3)$  is an element of  $R'_V$ .

For  $(i, k) \in I_{(12,j)} \times I_{(12,j')}$  with  $j < j'$  and  $\mu \in \mathbb{F}$ , define

$$\widetilde{g}_{i,k}(\mu) = \begin{cases} h_{(12)}(D) & \text{if } (j, j') \neq (2, 3), \\ g_{\sigma(i), \tau(k)}(\mu) h_{(12)}(D) & \text{if } (j, j') = (2, 3). \end{cases}$$

where  $D \in \text{GL}(U_{(12)}^{+1})$  is defined by  $De_k = e_k + \mu e_i$  and  $De_\ell = e_\ell$  for  $\ell \in I_{(12)} - \{k\}$ . Then the following lemma is also clear from the definition of  $U(s_7, s_4, s_6)$  and  $h_{(12)}(D)$ .

**Lemma 4.4.** (i)  $\widetilde{g}_{i,k}(\mu) \in R'_V$ .

(ii)  $\widetilde{g}_{i,k}(\mu) e_{\overline{\kappa(i)}} = e_{\overline{\kappa(i)}} - \mu e_{\overline{\kappa(k)}}$  and  $\widetilde{g}_{i,k}(\mu) e_\ell = e_\ell$  for  $\ell \in I_{(12)} - \{\overline{\kappa(i)}\}$ .

Put  $U_\# = U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^{+1} \oplus U_{(13)}^{+1} \oplus U_{(12)}^{+2}$ . The space  $U_{(12)}^{+2}$  is decomposed as

$$U_{(12)}^{+2} = U_{(12,4)}^{+2} \oplus U_{(12,3)}^{+2} \oplus U_{(12,2)}^{+2} \oplus U_{(12,1)}^{+2}$$

where  $U_{(12,j)}^{+2} = \bigoplus_{i \in I_{(12,j)}} \mathbb{F}e_{\overline{\kappa(i)}}$  for  $j = 1, 2, 3, 4$ . Let  $P$  denote the parabolic subgroup of  $\text{GL}(U_\#)$  stabilizing the flag

$$\begin{aligned} U_{(8,2,3)}^{+1} &\subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^{+1} \subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^{+1} \oplus U_{(13)}^{+1} \subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^{+1} \oplus U_{(13)}^{+1} \oplus U_{(12,4)}^{+2} \\ &\subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^{+1} \oplus U_{(13)}^{+1} \oplus U_{(12,4)}^{+2} \oplus U_{(12,3)}^{+2} \\ &\subset U_{(8,2,3)}^{+1} \oplus U_{(10,3)}^{+1} \oplus U_{(13)}^{+1} \oplus U_{(12,4)}^{+2} \oplus U_{(12,3)}^{+2} \oplus U_{(12,2)}^{+2}. \end{aligned}$$

**Lemma 4.5.** For any element  $p \in P$ , there exists a  $g \in R'_V$  such that  $g|_{U_\#} = p$  and that  $g$  acts trivially on  $U_{(15)}^{+1}$ .

*Proof.* We have only to construct  $g \in R'_V$  for the elements of a set of generators in  $P$ .

First consider the standard Levi subgroup

$$\mathrm{GL}(U_{(8,2,3)}^{+1}) \times \mathrm{GL}(U_{(10,3)}^+) \times \mathrm{GL}(U_{(13)}^+) \times \mathrm{GL}(U_{(12,4)}^{+2}) \times \mathrm{GL}(U_{(12,3)}^{+2}) \times \mathrm{GL}(U_{(12,2)}^{+2}) \times \mathrm{GL}(U_{(12,1)}^{+2})$$

of  $P$ . For  $A \in \mathrm{GL}(U_{(8,2,3)}^{+1})$ ,  $h_{(8)}(\tilde{A})$  is a desired element of  $R'_V$  where  $\tilde{A} \in \mathrm{GL}(U_{(8)}^{+1})$  is the canonical extension of  $A$ . Similarly for  $B \in \mathrm{GL}(U_{(10,3)}^+)$ ,  $h_{(10)}(\tilde{B})$  is a desired element of  $R'_V$ . For  $C \in \mathrm{GL}(U_{(13)}^+)$ ,  $h_{(13)}(C)$  is a desired element of  $R'_V$ .

For  $A_j \in \mathrm{GL}(U_{(12,j)}^{+2})$  with  $j = 1, 2, 3, 4$ , we can take  $B_j \in \mathrm{GL}(U_{(12,j)}^{+1})$  such that  $h_{(12)}(\tilde{B}_j)|_{U_{(12,j)}^{+2}} = A_j$ . Then  $\tilde{h}(B_j)$  are desired elements of  $R'_V$  by Lemma 4.3.

So we have only to consider the generators of the unipotent radical of  $P$ . For

$$(i, k) \in (I_{(8,3)} \times I_{(10,3)}) \sqcup (I_{(8,3)} \times I_{(13)}) \sqcup (I_{(10,3)} \times I_{(13)}) \\ \sqcup (I_{(8,3)} \times I_{(\overline{12})}) \sqcup (I_{(10,3)} \times I_{(\overline{12})}) \sqcup (I_{(13)} \times I_{(\overline{12})}),$$

we have  $g = g_{i,k}(\mu) \in R'_V$  ( $\mu \in \mathbb{F}$ ) such that  $ge_k = e_k + \mu e_i$  and that  $ge_\ell = e_\ell$  for  $\ell \in I_+ - \{k\}$  by Lemma 3.21 (i). For  $(i, k) \in I_{(12,j)} \times I_{(12,j')}$  with  $1 \leq j < j' \leq 4$  and  $\mu \in \mathbb{F}$ , we have  $\tilde{g}_{i,k}(-\mu) \in R'_V$  such that

$$\tilde{g}_{i,k}(-\mu)e_{\overline{\kappa(i)}} = e_{\overline{\kappa(i)}} + \mu e_{\overline{\kappa(k)}} \quad \text{and that} \quad \tilde{g}_{i,k}(-\mu)e_\ell = e_\ell \text{ for } \ell \in I_{(\overline{12})} - \{\overline{\kappa(i)}\}$$

by Lemma 4.4.  $\tilde{g}_{i,k}(-\mu)$  acts trivially on  $U_{(15)}^+$  by its construction. Thus we have constructed desired elements of  $R'_V$  for all the generators of the unipotent radical of  $P$ .  $\square$

**4.8. Finiteness of  $S_5 \oplus S_6$ -part.** Put  $H_{(15)} = (\mathrm{SO}(U_{(15)}) \cap R_V)|_{U_{(15)}^+}$ . Then we showed in [M13] that

$$H_{(15)} \cong \begin{cases} \mathrm{Sp}_{b_{15}}(\mathbb{F}) & \text{if } b_{15} \text{ is even and } \varepsilon = 0, \\ 1 \times \mathrm{Sp}_{b_{15}-1}(\mathbb{F}) & \text{if } b_{15} \text{ is odd,} \\ Q_{b_{15}} & \text{if } b_{15} \text{ is even and } \varepsilon = 1 \end{cases}$$

where  $Q_{b_{15}} = \{g \in \mathrm{Sp}_{b_{15}}(\mathbb{F}) \mid gv = v\}$  with some  $0 \neq v \in U_{(15)}^+$ . We showed in [M13] that there are a finite number of  $H_{(15)}$ -orbits on the full flag variety of  $\mathrm{GL}(U_{(15)}^+)$ .

It is clear that  $H_{(15)} \subset R'_V$ . Hence the restriction of  $R'_V$  to  $U_\# \oplus U_{(15)}^+$  contains the group  $P \times H_{(15)}$  by Lemma 4.5. So it contains a group of the form  $B \times H_{(15)}$  where  $B$  is a Borel subgroup of  $\mathrm{GL}(U_\#)$  contained in  $P$ . Applying Proposition 6.3, we have a finite number of  $R'_V$ -orbits on the Grassmann variety of  $U_\# \oplus U_{(15)}^+$ .

Thus we have proved that the triple flag variety  $\mathcal{T}_{(\alpha_1, \alpha_2), (\beta), (n)}$  is of finite type.

## 5. FINITENESS OF $\mathcal{T}_{(\alpha), (1), (1^n)}$

Let  $U_+$  and  $U_-$  be  $\alpha$ -dimensional and one-dimensional isotropic subspaces of  $\mathbb{F}^{2n+1}$ , respectively. Put  $R = P_{U_+} \cap P_{U_-}$ . Then we have only to consider  $R$ -orbit decomposition of the full flag variety  $M_0$  of  $G$ .

Since  $a_0 + a_- + a_1 = \beta = 1$ , there are three cases:

$$(i) a_0 = 1, \quad (ii) a_- = 1 \quad \text{and} \quad (iii) a_1 = 1.$$

If  $a_0 = 1$ , then  $U_- \subset U_+$ . So the decomposition is reduced to the Bruhat decomposition and hence the number of orbits is finite. Thus we have only to consider the remaining two cases.

**5.1. Case of  $a_- = 1$ .** As in Section 3, we put

$$U_+ = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_\alpha \quad \text{and} \quad U_- = \mathbb{F}e_{\alpha+1}.$$

Let  $V_1 \subset \cdots \subset V_n$  be a full flag in  $\mathbb{F}^{2n+1}$ . Then by Theorem 3.15, we may assume  $V_n = V = V(b_1, \dots, b_{15}, \varepsilon)$ . For each  $V$ , we have only to show that there are a finite number of  $R_V$ -orbits on the full flag variety  $M_0(V)$  of  $\text{GL}(V)$ .

Since  $a_0 = a_1 = 0$  and since

$$a_0 = b_1 + b_2, \quad a_1 = b_5 + b_9 + 2b_{12} + b_{15} + b_{13} + b_8 + b_6,$$

we have  $b_i = 0$  for  $i = 1, 2, 5, 6, 8, 9, 12, 13$  and  $15$ . Since

$$b_4 + b_7 + b_9 + b_{11} = a_- = 1,$$

there are three cases:

$$(a) b_4 = 1, \quad (b) b_7 = 1 \quad \text{and} \quad (c) b_{11} = 1.$$

**5.1.1. Case of  $b_4 = 1$ .**  $V$  is written as  $V = V_{(3)} \oplus V_{(4)} \oplus V_{(14)} \oplus V_{(10)}$  with

$$V_{(3)} = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, \quad V_{(4)} = \mathbb{F}e_{\alpha+1}, \quad V_{(14)} = \mathbb{F}e_{\alpha+2} \oplus \cdots \oplus \mathbb{F}e_n$$

$$\text{and} \quad V_{(10)} = \mathbb{F}e_{\bar{\alpha}} \oplus \cdots \oplus \mathbb{F}e_{\overline{b_3+1}}.$$

**Lemma 5.1.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_{\alpha+1}, e_{\alpha+2}, \dots, e_n, e_{\bar{\alpha}}, \dots, e_{\overline{b_3+1}}$$

*of  $V$ , the group  $R_V|_V$  is represented as*

$$R_V|_V = Q_{(4)} = \left\{ \begin{pmatrix} A & 0 & * & * \\ 0 & \lambda & * & * \\ 0 & 0 & B & * \\ 0 & 0 & 0 & C \end{pmatrix} \right\}$$

*where  $A \in \text{GL}_{b_3}(\mathbb{F})$ ,  $B \in \text{GL}_{n-\alpha-1}(\mathbb{F})$ ,  $C \in \text{GL}_{\alpha-b_3}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ .*

*Proof.* It is clear from the conditions on  $R_V$  that  $R_V|_V \subset Q_{(4)}$ .

Take elements  $h_{(3)}(A)$ ,  $h_{(4)}(\lambda)$ ,  $h_{(14)}(B)$  and  $h_{(10)}(C^*)$  constructed in Lemma 3.18 where  $C^* = J_{\alpha-b_3} {}^t C^{-1} J_{\alpha-b_3}$ . Then

$$h_{(3)}(A)h_{(4)}(\lambda)h_{(14)}(B)h_{(10)}(C^*)|_V = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & C \end{pmatrix}.$$

So we have only to construct generators of the unipotent part in the same way as in Lemma 3.21.

For  $i \in I_{(3)} \sqcup I_{(4)}$ ,  $k \in I_{(14)}$  and  $\mu \in \mathbb{F}$ , define  $g = g_{i,k}(\mu) \in G$  by

$$ge_k = e_k + \mu e_i, \quad ge_{\bar{i}} = e_{\bar{i}} - \mu e_{\bar{k}}$$

and  $ge_\ell = e_\ell$  for  $\ell \in I - \{k, \bar{i}\}$ . Then we see that  $g$  stabilizes  $U_+, U_-$  and  $V$ . So  $g \in R_V$ .

Similarly for  $i \in I_{(14)}$ ,  $k \in I_{(10)}$  and  $\mu \in \mathbb{F}$ , we can define  $g = g_{i,\bar{k}}(\mu) \in R_V$  by

$$ge_{\bar{k}} = e_{\bar{k}} + \mu e_i, \quad ge_{\bar{i}} = e_{\bar{i}} - \mu e_k$$

and  $ge_\ell = e_\ell$  for  $\ell \in I - \{\bar{k}, \bar{i}\}$ . We can construct the remaining generators as in Lemma 3.19.  $\square$

By Proposition 6.5 in the appendix,  $Q_{(4)}$ -orbit decomposition on  $M(V)$  is reduced to the orbit decomposition of the full flag variety of  $\mathrm{GL}_{b_3+1}(\mathbb{F})$  by the subgroup consisting of matrices of the form

$$\begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix}.$$

There are  $(b_3 + 2)(b_3 + 1)/2$  orbits in this decomposition ([M13]).

5.1.2. *Case of  $b_{11} = 1$ .*  $V$  is written as  $V = V_{(3)} \oplus V_{(14)} \oplus V_{(11)} \oplus V_{(10)}$  with

$$\begin{aligned} V_{(3)} &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, & V_{(14)} &= \mathbb{F}e_{\alpha+2} \oplus \cdots \oplus \mathbb{F}e_n, & V_{(11)} &= \mathbb{F}e_{\alpha+1} \\ \text{and } V_{(10)} &= \mathbb{F}e_{\bar{\alpha}} \oplus \cdots \oplus \mathbb{F}e_{\bar{b}_3+1}. \end{aligned}$$

We can prove the following lemma in the same way as Lemma 5.1.

**Lemma 5.2.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_{\alpha+2}, \dots, e_n, e_{\alpha+1}, e_{\bar{\alpha}}, \dots, e_{\bar{b}_3+1}$$

*of  $V$ , the group  $R_V|_V$  is represented as*

$$R_V|_V = Q_{(11)} = \left\{ \begin{pmatrix} A & * & * & * \\ 0 & B & * & * \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & C \end{pmatrix} \right\}$$

*where  $A \in \mathrm{GL}_{b_3}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{n-\alpha-1}(\mathbb{F})$ ,  $C \in \mathrm{GL}_{\alpha-b_3}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ .*

By Proposition 6.5,  $Q_{(11)}$ -orbit decomposition on  $M(V)$  is reduced to the orbit decomposition of the full flag variety of  $\mathrm{GL}_{\alpha-b_3+1}(\mathbb{F})$  by the subgroup consisting of matrices of the form  $\begin{pmatrix} \lambda & 0 \\ 0 & C \end{pmatrix}$  with  $(\alpha - b_3 + 2)(\alpha - b_3 + 1)/2$  orbits.

5.1.3. *Case of  $b_7 = 1$ .*  $V$  is written as  $V = V_{(3)} \oplus V_{(7)}^1 \oplus V_{(14)} \oplus V_{(10)} \oplus V_{(7)}^2$  with

$$\begin{aligned} V_{(3)} &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, & V_{(7)}^1 &= \mathbb{F}(e_{b_3+1} + e_{\alpha+1}), & V_{(14)} &= \mathbb{F}e_{\alpha+2} \oplus \cdots \oplus \mathbb{F}e_n, \\ V_{(10)} &= \mathbb{F}e_{\bar{\alpha}} \oplus \cdots \oplus \mathbb{F}e_{\bar{b}_3+2} & \text{and } V_{(7)}^2 &= \mathbb{F}(e_{\bar{b}_3+1} - e_{\alpha+1}). \end{aligned}$$



**Lemma 5.3.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_{b_3+1} + e_{\alpha+1}, e_{\alpha+2}, \dots, e_n, e_{\bar{\alpha}}, \dots, e_{\overline{b_3+2}}, e_{\overline{b_3+1}} - e_{\overline{\alpha+1}}$$

of  $V$ , the group  $R_V|_V$  is represented as

$$R_V|_V = Q_{(7)} = \left\{ \begin{pmatrix} A & * & * & * & * \\ 0 & \lambda & * & * & * \\ 0 & 0 & B & * & * \\ 0 & 0 & 0 & C & * \\ 0 & 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \right\}$$

where  $A \in \mathrm{GL}_{b_3}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{n-\alpha-1}(\mathbb{F})$ ,  $C \in \mathrm{GL}_{\alpha-b_3-1}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ .

*Proof.* Since  $R_V$  stabilizes subspaces

$$\begin{aligned} V \cap U_+ &= V_{(3)}, \quad V \cap (U_+ \oplus U_-) = V_{(3)} \oplus V_{(7)}^1, \\ V \cap (U_+ \oplus U_-)^\perp &= V_{(3)} \oplus V_{(7)}^1 \oplus V_{(14)} \\ \text{and } V \cap U_-^\perp &= V_{(3)} \oplus V_{(7)}^1 \oplus V_{(14)} \oplus V_{(10)} \end{aligned}$$

of  $V$ ,  $R_V|_V$  is contained in the parabolic subgroup

$$\left\{ \begin{pmatrix} A & * & * & * & * \\ 0 & \lambda & * & * & * \\ 0 & 0 & B & * & * \\ 0 & 0 & 0 & C & * \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix} \right\}$$

where  $A \in \mathrm{GL}_{b_3}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{n-\alpha-1}(\mathbb{F})$ ,  $C \in \mathrm{GL}_{\alpha-b_3-1}(\mathbb{F})$  and  $\lambda, \mu \in \mathbb{F}^\times$ . Let  $g$  be an element of  $R_V$ . Since  $gU_- = U_-$ , we have  $ge_{\alpha+1} = \lambda e_{\alpha+1}$  with some  $\lambda \in \mathbb{F}^\times$ . Hence

$$g(e_{b_3+1} + e_{\alpha+1}) \in \lambda(e_{b_3+1} + e_{\alpha+1}) + V_{(3)}.$$

On the other hand, we have

$$ge_{\overline{\alpha+1}} = \lambda^{-1}e_{\overline{\alpha+1}} + \bigoplus_{i \neq \overline{\alpha+1}} \mathbb{F}e_i$$

since  $g$  preserves the bilinear form  $(\ , \ )$ . Hence

$$g(e_{\overline{b_3+1}} - e_{\overline{\alpha+1}}) \in \lambda^{-1}(e_{\overline{b_3+1}} - e_{\overline{\alpha+1}}) + (V_{(3)} \oplus V_{(7)}^1 \oplus V_{(14)} \oplus V_{(10)}).$$

Thus we have  $R_V|_V \subset Q_{(7)}$ .

As in the proof of Lemma 5.1, the elements

$$h_{(3)}(A)h_{(7)}(\lambda)h_{(14)}(B)h_{(10)}(C^*)|_V = \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}$$

are contained in  $R_V|_V$  for  $A \in \mathrm{GL}_{b_3}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{n-\alpha-1}(\mathbb{F})$ ,  $C \in \mathrm{GL}_{\alpha-b_3-1}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ . So we have only to consider the generators of the unipotent part.

For  $i \in I_{(3)}$  and  $\mu \in \mathbb{F}$ , define  $g = g_{i,b_3+1}(\mu) \in G$  by

$$ge_{b_3+1} = e_{b_3+1} + \mu e_i, \quad ge_{\bar{i}} = e_{\bar{i}} - \mu e_{\overline{b_3+1}}$$

and  $ge_\ell = e_\ell$  for  $\ell \in I - \{\alpha + 1, \bar{i}\}$ . Then we see that  $g$  stabilizes  $U_+, U_-$  and  $V$ . So  $g \in R_V$ .

For  $k \in I_{(14)}$  and  $\mu \in \mathbb{F}$ , define  $g = g_{b_3+1,k}(\mu) \in G$  by

$$ge_k = e_k + \mu(e_{b_3+1} + e_{\alpha+1}), \quad ge_{\overline{b_3+1}} = e_{\overline{b_3+1}} - \mu e_{\bar{k}}, \quad ge_{\overline{\alpha+1}} = e_{\overline{\alpha+1}} - \mu e_{\bar{k}}$$

and  $ge_\ell = e_\ell$  for  $\ell \in I - \{k, \overline{b_3+1}, \overline{\alpha+1}\}$ . Then we see that  $g \in R_V$ .

For  $i \in I_{(14)}$ ,  $k \in I_{(10)}$  and  $\mu \in \mathbb{F}$ , define  $g = g_{i,\bar{k}}(\mu) \in R_V$  by

$$ge_{\bar{k}} = e_{\bar{k}} + \mu e_i, \quad ge_{\bar{i}} = e_{\bar{i}} - \mu e_k$$

and  $ge_\ell = e_\ell$  for  $\ell \in I - \{\bar{k}, \bar{i}\}$ .

For  $i \in I_{(10)}$  and  $\mu \in \mathbb{F}$ , define  $g = g_{\bar{i},\overline{b_3+1}}(\mu) \in R_V$  by

$$ge_{\overline{b_3+1}} = e_{\overline{b_3+1}} + \mu e_{\bar{i}}, \quad ge_i = e_i - \mu e_{b_3+1}$$

and  $ge_\ell = e_\ell$  for  $\ell \in I - \{i, \overline{b_3+1}\}$ . We can construct the remaining generators as in Lemma 3.19.  $\square$

By this lemma,  $R_V$ -orbit decomposition of  $M_0(V)$  is reduced to the Bruhat decomposition. So there are a finite number of  $R_V$ -orbits on  $M_0(V)$ .

**5.2. Case of  $a_1 = 1$ .** As in Section 3, we put

$$U_+ = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_\alpha \quad \text{and} \quad U_- = \mathbb{F}e_{\bar{\alpha}} = \mathbb{F}e_{2n+2-\alpha}.$$

Since  $a_0 = a_- = 0$  and since

$$a_0 = b_1 + b_2, \quad a_- = b_4 + b_7 + b_9 + b_{11},$$

we have  $b_1 = b_2 = b_4 = b_7 = b_9 = b_{11} = 0$ . Since

$$a_1 = b_5 + b_9 + 2b_{12} + b_{15} + b_{13} + b_8 + b_6 = 1,$$

there are five cases:

$$(a) \ b_5 = 1, \quad (b) \ b_{15} = 1, \quad (c) \ b_{13} = 1, \quad (d) \ b_8 = 1 \quad \text{and} \quad (e) \ b_6 = 1.$$

**5.2.1. Case of  $b_5 = 1$ .**  $V$  is written as  $V = V_{(3)} \oplus V_{(5)} \oplus V_{(14)} \oplus V_{(10)}$  with

$$V_{(3)} = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, \quad V_{(5)} = \mathbb{F}e_\alpha, \quad V_{(14)} = \mathbb{F}e_{\alpha+1} \oplus \cdots \oplus \mathbb{F}e_n$$

$$\text{and} \quad V_{(10)} = \mathbb{F}e_{\overline{\alpha-1}} \oplus \cdots \oplus \mathbb{F}e_{\overline{b_3+1}}.$$

We can easily prove:

**Lemma 5.4.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_\alpha, e_{\alpha+1}, \dots, e_n, e_{\overline{\alpha-1}}, \dots, e_{\overline{b_3+1}}$$

of  $V$ ,  $R_V|_V$  is represented as

$$R_V|_V = Q_{(5)} = \left\{ \begin{pmatrix} A & * & * & * \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & B & * \\ 0 & 0 & 0 & C \end{pmatrix} \right\}$$

where  $A \in \mathrm{GL}_{b_3}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{n-\alpha}(\mathbb{F})$ ,  $C \in \mathrm{GL}_{\alpha-b_3-1}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ .

Applying Proposition 6.7 in the appendix, we see that there are a finite number of  $R_V$ -orbits on  $M_0(V)$ .

5.2.2. *Case of  $b_6 = 1$ .*  $V$  is written as  $V = V_{(3)} \oplus V_{(14)} \oplus V_{(6)} \oplus V_{(10)}$  with

$$\begin{aligned} V_{(3)} &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, & V_{(14)} &= \mathbb{F}e_{\alpha+1} \oplus \cdots \oplus \mathbb{F}e_n, & V_{(6)} &= \mathbb{F}e_{\bar{\alpha}} \\ \text{and } V_{(10)} &= \mathbb{F}e_{\overline{\alpha-1}} \oplus \cdots \oplus \mathbb{F}e_{\overline{b_3+1}}. \end{aligned}$$

We can easily prove:

**Lemma 5.5.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_{\alpha+1}, \dots, e_n, e_{\bar{\alpha}}, e_{\overline{\alpha-1}}, \dots, e_{\overline{b_3+1}}$$

of  $V$ ,  $R_V|_V$  is represented as

$$R_V|_V = Q_{(6)} = \left\{ \begin{pmatrix} A & * & 0 & * \\ 0 & B & 0 & * \\ 0 & 0 & \lambda & * \\ 0 & 0 & 0 & C \end{pmatrix} \right\}$$

where  $A \in \mathrm{GL}_{b_3}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{n-\alpha}(\mathbb{F})$ ,  $C \in \mathrm{GL}_{\alpha-b_3-1}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ .

Applying Proposition 6.7, we see that there are a finite number of  $R_V$ -orbits on  $M_0(V)$ .

5.2.3. *Case of  $b_{15} = 1$ .*  $V$  is written as  $V = V_{(3)} \oplus V_{(14)} \oplus V_{(15)} \oplus V_{(10)}$  with

$$\begin{aligned} V_{(3)} &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, & V_{(14)} &= \mathbb{F}e_{\alpha+1} \oplus \cdots \oplus \mathbb{F}e_n, \\ V_{(15)} &= \mathbb{F}(e_{\bar{\alpha}} - \frac{1}{2}e_{\alpha} + e_{n+1}) & \text{and } V_{(10)} &= \mathbb{F}e_{\overline{\alpha-1}} \oplus \cdots \oplus \mathbb{F}e_{\overline{b_3+1}}. \end{aligned}$$

We can easily prove:

**Lemma 5.6.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_{\alpha+1}, \dots, e_n, e_{\bar{\alpha}} - \frac{1}{2}e_{\alpha} + e_{n+1}, e_{\overline{\alpha-1}}, \dots, e_{\overline{b_3+1}}$$

of  $V$ ,  $R_V|_V$  is represented as

$$R_V|_V = Q_{(15)} = \left\{ \begin{pmatrix} A & * & * & * \\ 0 & B & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & C \end{pmatrix} \right\}$$

where  $A \in \mathrm{GL}_{b_3}(\mathbb{F})$ ,  $B \in \mathrm{GL}_{n-\alpha}(\mathbb{F})$  and  $C \in \mathrm{GL}_{\alpha-b_3-1}(\mathbb{F})$ .

By Proposition 6.5,  $Q_{(15)}$ -orbit decomposition on  $M(V)$  is reduced to the orbit decomposition of the full flag variety of  $\mathrm{GL}_{\alpha-b_3}(\mathbb{F})$  by the subgroup consisting of matrices of the form  $\begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$  with  $(\alpha - b_3 + 1)(\alpha - b_3)/2$  orbits.

5.2.4. *Case of  $b_8 = 1$ .*  $V$  is written as  $V = V_{(3)} \oplus V_{(8)}^1 \oplus V_{(14)} \oplus V_{(10)} \oplus V_{(8)}^2$  with

$$\begin{aligned} V_{(3)} &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, & V_{(8)}^1 &= \mathbb{F}(e_{b_3+1} + e_{\bar{\alpha}}), & V_{(14)} &= \mathbb{F}e_{\alpha+1} \oplus \cdots \oplus \mathbb{F}e_n, \\ V_{(10)} &= \mathbb{F}e_{\overline{\alpha-1}} \oplus \cdots \oplus \mathbb{F}e_{\overline{b_3+2}} & \text{and} & & V_{(8)}^2 &= \mathbb{F}(e_{\overline{b_3+1}} - e_{\alpha}). \end{aligned}$$

**Lemma 5.7.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_{b_3+1} + e_{\bar{\alpha}}, e_{\alpha+1}, \dots, e_n, e_{\overline{\alpha-1}}, \dots, e_{\overline{b_3+2}}, e_{\overline{b_3+1}} - e_{\alpha}$$

*of  $V$ ,  $R_V|_V$  is represented as*

$$R_V|_V = Q_{(8)} = \left\{ \begin{pmatrix} A & * & * & * & * \\ 0 & \lambda & 0 & * & * \\ 0 & 0 & B & * & * \\ 0 & 0 & 0 & C & * \\ 0 & 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \right\}$$

where  $A \in \text{GL}_{b_3}(\mathbb{F})$ ,  $B \in \text{GL}_{n-\alpha}(\mathbb{F})$ ,  $C \in \text{GL}_{\alpha-b_3-2}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ .

*Proof.* Since  $R_V$  stabilizes subspaces

$$\begin{aligned} V \cap U_+ &= V_{(3)}, & V \cap (U_+ \oplus U_-) &= V_{(3)} \oplus V_{(8)}^1, \\ V \cap (U_+ \oplus U_-)^\perp &= V_{(3)} \oplus V_{(14)} & \text{and} & & V \cap U_-^\perp &= V_{(3)} \oplus V_{(8)}^1 \oplus V_{(14)} \oplus V_{(10)} \end{aligned}$$

of  $V$ ,  $R_V|_V$  is contained in the group

$$\left\{ \begin{pmatrix} A & * & * & * & * \\ 0 & \lambda & 0 & * & * \\ 0 & 0 & B & * & * \\ 0 & 0 & 0 & C & * \\ 0 & 0 & 0 & 0 & \mu \end{pmatrix} \right\}$$

where  $A \in \text{GL}_{b_3}(\mathbb{F})$ ,  $B \in \text{GL}_{n-\alpha}(\mathbb{F})$ ,  $C \in \text{GL}_{\alpha-b_3-2}(\mathbb{F})$  and  $\lambda, \mu \in \mathbb{F}^\times$ . Let  $g$  be an element of  $R_V$ . Since  $gU_- = U_-$ , we have  $ge_{\bar{\alpha}} = \lambda e_{\bar{\alpha}}$  with some  $\lambda \in \mathbb{F}^\times$ . Hence

$$g(e_{b_3+1} + e_{\bar{\alpha}}) \in \lambda(e_{b_3+1} + e_{\bar{\alpha}}) + V_{(3)}.$$

On the other hand, we have

$$ge_{\alpha} = \lambda^{-1}e_{\alpha} + \bigoplus_{i \neq \alpha} \mathbb{F}e_i$$

since  $g$  preserves the bilinear form  $(\ , \ )$ . Hence

$$g(e_{\overline{b_3+1}} - e_{\alpha}) \in \lambda^{-1}(e_{\overline{b_3+1}} - e_{\alpha}) + (V_{(3)} \oplus V_{(8)}^1 \oplus V_{(14)} \oplus V_{(10)}).$$

Thus we have  $R_V|_V \subset Q_{(8)}$ .

The proof of  $Q_{(8)} \subset R_V|_V$  is similar to Lemma 5.3. □

By Proposition 6.5,  $Q_{(8)}$ -orbit decomposition on  $M(V)$  is reduced to the orbit decomposition of the full flag variety of  $\text{GL}_{n-\alpha+1}(\mathbb{F})$  by the subgroup consisting of matrices of the form  $\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$  with  $(n - \alpha + 2)(n - \alpha + 1)/2$  orbits.

5.2.5. *Case of  $b_{13} = 1$ .*  $V$  is written as  $V = V_{(3)} \oplus V_{(14)} \oplus V_{(13)}^1 \oplus V_{(13)}^2 \oplus V_{(10)}$  with

$$\begin{aligned} V_{(3)} &= \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{b_3}, & V_{(14)} &= \mathbb{F}e_{\alpha+1} \oplus \cdots \oplus \mathbb{F}e_{n-1}, & V_{(13)}^1 &= \mathbb{F}(e_\alpha + e_n), \\ V_{(13)}^2 &= \mathbb{F}(e_{\bar{\alpha}} - e_{n+2}) & \text{and} & & V_{(10)} &= \mathbb{F}e_{\alpha-1} \oplus \cdots \oplus \mathbb{F}e_{b_3+1}. \end{aligned}$$

**Lemma 5.8.** *With respect to the basis*

$$e_1, \dots, e_{b_3}, e_{\alpha+1}, \dots, e_n, e_\alpha + e_n, e_{\bar{\alpha}} - e_{n+2}, e_{\alpha-1}, \dots, e_{b_3+1}$$

*of  $V$ ,  $R_V|_V$  is represented as*

$$R_V|_V = Q_{(13)} = \left\{ \begin{pmatrix} A & * & * & * & * \\ 0 & B & * & * & * \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda^{-1} & * \\ 0 & 0 & 0 & 0 & C \end{pmatrix} \right\}$$

where  $A \in \text{GL}_{b_3}(\mathbb{F})$ ,  $B \in \text{GL}_{n-\alpha-1}(\mathbb{F})$ ,  $C \in \text{GL}_{\alpha-b_3-1}(\mathbb{F})$  and  $\lambda \in \mathbb{F}^\times$ .

*Proof.* Since  $R_V$  stabilizes subspaces

$$\begin{aligned} V \cap U_+ &= V_{(3)}, & V \cap (U_+ \oplus U_-)^\perp &= V_{(3)} \oplus V_{(14)}, \\ V \cap U_+^\perp &= V_{(3)} \oplus V_{(14)} \oplus V_{(13)}^1, & V \cap (W_+ \oplus U_-)^\perp &= V_{(3)} \oplus V_{(14)} \oplus V_{(13)}^2 \\ \text{and } V \cap U_-^\perp &= V_{(3)} \oplus V_{(14)} \oplus V_{(13)}^2 \oplus V_{(10)} \end{aligned}$$

of  $V$ ,  $R_V|_V$  is contained in the group

$$\left\{ \begin{pmatrix} A & * & * & * & * \\ 0 & B & * & * & * \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \mu & * \\ 0 & 0 & 0 & 0 & C \end{pmatrix} \right\}$$

where  $A \in \text{GL}_{b_3}(\mathbb{F})$ ,  $B \in \text{GL}_{n-\alpha-1}(\mathbb{F})$ ,  $C \in \text{GL}_{\alpha-b_3-1}(\mathbb{F})$  and  $\lambda, \mu \in \mathbb{F}^\times$ . Let  $g$  be an element of  $R_V$ . Since  $gU_- = U_-$ , we have  $ge_{\bar{\alpha}} = \lambda^{-1}e_{\bar{\alpha}}$  with some  $\lambda \in \mathbb{F}^\times$ . Hence

$$g(e_{\bar{\alpha}} - e_{n+2}) \in \lambda^{-1}(e_{\bar{\alpha}} - e_{n+2}) + (V_{(3)} \oplus V_{(14)}).$$

On the other hand, we have

$$ge_\alpha = \lambda e_\alpha + \bigoplus_{i \neq \alpha} \mathbb{F}e_i$$

since  $g$  preserves the bilinear form  $(\ , \ )$ . Hence

$$g(e_\alpha + e_n) \in \lambda(e_\alpha + e_n) + (V_{(3)} \oplus V_{(14)}).$$

Thus we have  $R_V|_V \subset Q_{(13)}$ .

The proof of  $Q_{(13)} \subset R_V|_V$  is similar to Lemma 5.3. □

By Proposition 6.5,  $Q_{(13)}$ -orbit decomposition on  $M(V)$  is reduced to the orbit decomposition of the full flag variety  $M'$  of  $\mathrm{GL}_{\alpha-b_3+1}(\mathbb{F})$  by the subgroup  $H$  consisting of matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & * \\ 0 & 0 & C \end{pmatrix}.$$

Let  $\tilde{H}$  denote the subgroup of  $\mathrm{GL}_{\alpha-b_3+1}(\mathbb{F})$  consisting of matrices of the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & * \\ 0 & 0 & C \end{pmatrix}$$

where  $\lambda, \mu \in \mathbb{F}^\times$  and  $C \in \mathrm{GL}_{\alpha-b_3-1}(\mathbb{F})$ . Then there are a finite number of  $\tilde{H}$ -orbits on  $M'$  by Proposition 6.7. Since the center  $Z = \{\lambda I_{\alpha-b_3+1} \mid \lambda \in \mathbb{F}^\times\}$  acts trivially on  $M'$  and since

$$|\tilde{H}/ZH| = \begin{cases} 2 & \text{if } (\mathbb{F}^\times)^2 \neq \mathbb{F}^\times \\ 1 & \text{if } (\mathbb{F}^\times)^2 = \mathbb{F}^\times, \end{cases}$$

there are a finite number of  $H$ -orbits on  $M'$ .

*Remark 5.9.* If  $(\mathbb{F}^\times)^2 \neq \mathbb{F}^\times$ , then some  $\tilde{H}$ -orbits on  $M'$  split into two  $H$ -orbits. For example,  $P^1(\mathbb{R})$  has two open  $H$ -orbits where

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mid \lambda \in \mathbb{R}^\times \right\}.$$

Thus we have completed the proof of finiteness of the triple flag variety  $\mathcal{T}_{(\alpha),(1),(1^n)}$ .

## 6. APPENDIX

### 6.1. An elementary lemma for $\mathrm{GL}_n(\mathbb{F})$ .

**Lemma 6.1.** *Let  $U, U', V$  and  $V'$  be subspaces of  $\mathbb{F}^n$  over an arbitrary field  $\mathbb{F}$  such that  $\dim U = \dim U'$ ,  $\dim V = \dim V'$  and that  $\dim(U \cap V) = \dim(U' \cap V')$ . Then there exists an element  $g$  of  $G = \mathrm{GL}_n(\mathbb{F})$  such that*

$$gU = U' \quad \text{and that} \quad gV = V'.$$

*Proof.* Put  $\dim U = \dim U' = k + m$ ,  $\dim V = \dim V' = \ell + m$  and  $\dim(U \cap V) = \dim(U' \cap V') = m$ . Let  $w_1, \dots, w_m$  and  $w'_1, \dots, w'_m$  be bases of  $W = U \cap V$  and  $W' = U' \cap V'$ , respectively. Then we can take vectors  $u_1, \dots, u_k, u'_1, \dots, u'_k, v_1, \dots, v_\ell, v'_1, \dots, v'_\ell$  such that

$$U = \mathbb{F}u_1 \oplus \dots \oplus \mathbb{F}u_k \oplus W, \quad U' = \mathbb{F}u'_1 \oplus \dots \oplus \mathbb{F}u'_k \oplus W',$$

$$V = \mathbb{F}v_1 \oplus \dots \oplus \mathbb{F}v_\ell \oplus W \quad \text{and that} \quad V' = \mathbb{F}v'_1 \oplus \dots \oplus \mathbb{F}v'_\ell \oplus W'.$$

Since the vectors  $u_1, \dots, u_k, v_1, \dots, v_\ell, w_1, \dots, w_m$  and  $u'_1, \dots, u'_k, v'_1, \dots, v'_\ell, w'_1, \dots, w'_m$  are linearly independent, we can take an element  $g \in G$  such that

$$gu_1 = u'_1, \dots, gu_k = u'_k, \quad gv_1 = v'_1, \dots, gv_\ell = v'_\ell, \quad gw_1 = w'_1, \dots, gw_m = w'_m.$$

This  $g$  is a desired element of  $G$ .  $\square$

### 6.2. Finiteness of some orbit decomposition on the Grassmann variety.

Consider the direct sum decomposition  $\mathbb{F}^{m+n} = U_1 \oplus U_2$  with  $U_1 = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_m$  and  $U_2 = \mathbb{F}e_{m+1} \oplus \cdots \oplus \mathbb{F}e_{m+n}$  over an arbitrary field  $\mathbb{F}$ . Write  $G_1 = \text{GL}(U_1)$  and  $G_2 = \text{GL}(U_2)$ . For  $0 < s < m+n$ , let  $\text{Gr}$  denote the Grassmann variety

$$\text{Gr} = \{s\text{-dimensional subspaces in } U_1 \oplus U_2\}.$$

Let  $\pi_i$  denote the projection  $U_1 \oplus U_2 \rightarrow U_i$  for  $i = 1, 2$ . Let  $S$  be an  $s$ -dimensional subspace of  $U_1 \oplus U_2$ . Then the  $G_1 \times G_2$ -orbit of  $S$  is determined by the four invariants

$$\dim(U_1 \cap S), \quad \dim \pi_1(S), \quad \dim(U_2 \cap S) \quad \text{and} \quad \dim \pi_2(S)$$

with  $\dim(U_1 \cap S) + \dim \pi_2(S) = \dim(U_2 \cap S) + \dim \pi_1(S) = s$ . Write  $p = \dim(U_1 \cap S)$ ,  $q = \dim(U_2 \cap S)$  and  $r = s - p - q$ . Then

$$\dim \pi_1(S) - p = \dim \pi_2(S) - q = r.$$

Consider the canonical full flag

$$U_{2,1} \subset \cdots \subset U_{2,n-1}$$

in  $U_2$  where  $U_{2,i} = \mathbb{F}e_{m+1} \oplus \cdots \oplus \mathbb{F}e_{m+i}$ . Let  $B_2$  denote the Borel subgroup of  $G_2$  stabilizing this full flag. Define subsets

$$\begin{aligned} J &= \{i_1, \dots, i_q\} = \{i \in I_2 \mid \dim(U_{2,i} \cap S) = \dim(U_{2,i-1} \cap S) + 1\} \\ \text{and } K &= \{j_1, \dots, j_r\} = \{i \in I_2 \mid \dim(U_{2,i} \cap S) = \dim(U_{2,i-1} \cap S), \\ &\quad \dim(\pi_2(S) \cap U_{2,i}) = \dim(\pi_2(S) \cap U_{2,i-1}) + 1\} \end{aligned}$$

of  $I_2 = \{m+1, \dots, m+n\}$  with  $i_1 < \cdots < i_q$  and  $j_1 < \cdots < j_r$ . Then we can take a  $b \in B_2$  such that

$$b\pi_2(S) = \mathbb{F}e_{i_1} \oplus \cdots \oplus \mathbb{F}e_{i_q} \oplus \mathbb{F}e_{j_1} \oplus \cdots \oplus \mathbb{F}e_{j_r}.$$

We can write

$$bS = \mathbb{F}v_1 \oplus \cdots \oplus \mathbb{F}v_p \oplus \mathbb{F}e_{i_1} \oplus \cdots \oplus \mathbb{F}e_{i_q} \oplus \mathbb{F}(w_1 + e_{j_1}) \oplus \cdots \oplus \mathbb{F}(w_r + e_{j_r})$$

with some linearly independent vectors  $v_1, \dots, v_p, w_1, \dots, w_r$  in  $U_1$ . So we can take a  $g_1 \in G_1$  such that  $S_0 = g_1 bS$  is of the form

$$(6.1) \quad S_0 = U_{1,p} \oplus \mathbb{F}e_{i_1} \oplus \cdots \oplus \mathbb{F}e_{i_q} \oplus \mathbb{F}(e_{p+1} + e_{j_1}) \oplus \cdots \oplus \mathbb{F}(e_{p+r} + e_{j_r})$$

where  $U_{1,k} = \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_k$  for  $k = 1, \dots, m$

Let  $Q$  denote the isotropy subgroup

$$Q = \{g \in G_1 \times B_2 \mid gS_0 = S_0\}$$

of  $S_0$  in  $G_1 \times B_2$ . Let  $\tilde{\pi}_1 : G_1 \times G_2 \rightarrow G_1$  denote the projection.

**Lemma 6.2.**  $\tilde{\pi}_1(Q)$  is equal to the parabolic subgroup  $P_1$  of  $G_1$  stabilizing the flag  $U_{1,p} \subset U_{1,p+1} \subset \cdots \subset U_{1,p+r}$  in  $U_1$ .

*Proof.* Let  $g$  be an element of  $P_1$ . Then for  $k = 1, \dots, r$  we can write

$$ge_{p+k} \in \left( \sum_{i=1}^k a_{i,k} e_{p+i} \right) + U_{1,p}$$

with some  $a_{i,k} \in \mathbb{F}$ . Define an element  $b \in B_2$  by

$$be_{j_k} = \sum_{i=1}^k a_{i,k} e_{j_k} \quad \text{for } k = 1, \dots, r$$

and  $be_\ell = e_\ell$  for  $\ell \in I_2 - K$ . Then we have

$$bgS_0 = S_0$$

and therefore  $bg \in Q$ . Hence  $g \in \tilde{\pi}_1(Q)$ .

Conversely let  $g$  be an element of  $Q$ . Since  $g$  stabilizes  $U_1 \oplus U_{2,k}$  for  $k = 0, \dots, n$ , it stabilizes

$$S_0 \cap (U_1 \oplus U_{2,k}) = U_{1,p} \oplus \mathbb{F}e_{i_1} \oplus \dots \oplus \mathbb{F}e_{i_{k(1)}} \oplus \mathbb{F}(e_{p+1} + e_{j_1}) \oplus \dots \oplus \mathbb{F}(e_{p+k(2)} + e_{j_{k(2)}})$$

where  $i_{k(1)} \leq k < i_{k(1)+1}$  and  $j_{k(2)} \leq k < j_{k(2)+1}$ . Hence it stabilizes

$$\pi_1(S_0 \cap (U_1 \oplus U_{2,k})) = U_{1,p} \oplus \mathbb{F}e_{p+1} \oplus \dots \oplus \mathbb{F}e_{p+k(2)} = U_{1,p+k(2)}.$$

Since  $k(2)$  varies from 0 to  $r$ , we have  $\tilde{\pi}_1(g) \in P_1$ . □

**Proposition 6.3.** *Let  $H_1$  be a subgroup of  $G_1$  such that*

$$|H_1 \backslash G_1 / B_1| < \infty$$

*where  $B_1$  is a Borel subgroup of  $G_1$ . Then  $\text{Gr}$  consists of finitely many  $H_1 \times B_2$ -orbits.*

*Proof.* Since every  $G_1 \times B_2$ -orbit in  $\text{Gr}$  contains an element  $S_0$  of the form (6.1), we have only to show that  $(G_1 \times B_2)S_0 \cong (G_1 \times B_2)/Q$  is decomposed into a finite number of  $H_1 \times B_2$ -orbits. By the map  $\tilde{\pi}_1$ , we have

$$(H_1 \times B_2) \backslash (G_1 \times B_2) / Q \xrightarrow{\sim} H_1 \backslash G_1 / \tilde{\pi}_1(Q).$$

So the assertion follows from the assumption  $|H_1 \backslash G_1 / B_1| < \infty$  and Lemma 6.2. □

**6.3. An orbit decomposition of  $\text{GL}_n(\mathbb{F})/B$ .** Let  $V_1 \subset V_2 \subset \dots \subset V_{n-1}$  be the canonical full flag in  $\mathbb{F}^n$  defined by

$$V_1 = \mathbb{F}e_1, \quad V_2 = \mathbb{F}e_1 \oplus \mathbb{F}e_2, \quad \dots, \quad V_{n-1} = \mathbb{F}e_1 \oplus \dots \oplus \mathbb{F}e_{n-1}.$$

Then

$$B = \{g \in G \mid gV_i = V_i \text{ for } i = 1, \dots, n-1\} = \{\text{upper triangular matrices in } G\}$$

is a Borel subgroup of  $G = \text{GL}_n(\mathbb{F})$ .

Suppose  $n = \alpha_1 + \dots + \alpha_p$  with positive integers  $\alpha_1, \dots, \alpha_p$ . Define a partition  $I = I_1 \sqcup \dots \sqcup I_p$  of  $I = \{1, \dots, n\}$  by

$$I_1 = \{1, \dots, \alpha_1\}, \quad I_2 = \{\alpha_1 + 1, \dots, \alpha_1 + \alpha_2\}, \quad \dots, \quad I_p = \{\alpha_1 + \dots + \alpha_{p-1} + 1, \dots, n\}.$$



Put  $U_j = \oplus_{k \in I_j} \mathbb{F}e_k$ . Then we have a direct sum decomposition  $\mathbb{F}^n = U_1 \oplus \cdots \oplus U_p$ . Let  $P$  be the parabolic subgroup of  $G$  defined by

$$P = \left\{ \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_p \end{pmatrix} \mid A_k \in \mathrm{GL}_{\alpha_k}(\mathbb{F}) \text{ for } k = 1, \dots, p \right\}.$$

Then  $P$  is the isotropy subgroup in  $G$  for the flag  $U_1 \subset U_1 \oplus U_2 \subset \cdots \subset U_1 \oplus \cdots \oplus U_{p-1}$ .

Let  $S_n$  denote the symmetric group for  $I$ . For an element  $\sigma \in S_n$ , there corresponds a permutation matrix  $w_\sigma$  defined by

$$w_\sigma(e_i) = e_{\sigma(i)} \quad \text{for } i \in I.$$

Fix a  $j \in \{1, \dots, p\}$  and  $\sigma \in S_n$ . Then there exists a unique sequence  $r_1 < \cdots < r_{\alpha_j}$  such that

$$(\sigma(r_1), \dots, \sigma(r_{\alpha_j})) = I_j.$$

Consider the permutation  $\tau$  of  $I_j$  given by

$$(6.2) \quad \tau\sigma(r_1) = \beta + 1, \dots, \tau\sigma(r_{\alpha_j}) = \beta + \alpha_j$$

where  $\beta = \alpha_1 + \cdots + \alpha_{j-1}$ . Since  $w_\tau \in P$ , we have

$$Pw_\sigma B = Pw_\tau w_\sigma B = PwB$$

where  $w = w_\tau w_\sigma$ . It follows from (6.2) that

$$(6.3) \quad U_j \cap wV_{r_k} = \mathbb{F}e_{\beta+1} \oplus \cdots \oplus \mathbb{F}e_{\beta+k} \quad \text{for } k = 1, \dots, \alpha_j.$$

Let  $L_j$  be the subgroup of  $P$  defined by

$$L_j = \left\{ \begin{pmatrix} I_\beta & & 0 \\ & A_j & \\ 0 & & I_\gamma \end{pmatrix} \mid A_j \in \mathrm{GL}_{\alpha_j}(\mathbb{F}) \right\}$$

where  $\gamma = \alpha_{j+1} + \cdots + \alpha_p$  and  $N_j$  the normal subgroup of  $P$  defined by

$$N_j = \left\{ \begin{pmatrix} A_1 & & & & * \\ & \ddots & & & \\ & & A_{j-1} & & \\ & & & I_{\alpha_j} & \\ & & & & A_{j+1} \\ & & & & & \ddots \\ 0 & & & & & & A_p \end{pmatrix} \mid A_k \in \mathrm{GL}_{\alpha_k}(\mathbb{F}) \right\}.$$

Then  $P = L_j N_j = N_j L_j$ .

**Lemma 6.4.** (i)  $L_j \cap wBw^{-1} = L_j \cap B$ .

(ii)  $P \cap wBw^{-1} = (L_j \cap B)(N_j \cap wBw^{-1})$ .

*Proof.* (i) Let  $g$  be an element of  $L_j$ . Then  $g \in wBw^{-1}$  if and only if  $g(U_j \cap wV_k) = U_j \cap wV_k$  for  $k = 1, \dots, n-1$  since

$$wV_k = (U_1 \cap wV_k) \oplus \cdots \oplus (U_p \cap wV_k).$$

Hence the assertion follows from (6.3).

(ii) Suppose  $g \in P \cap wBw^{-1}$ . Then by (6.3), we have

$$ge_{\beta+k} \in (U_1 \oplus \cdots \oplus U_j) \cap wV_{r_k} = (U_1 \cap wV_{r_k}) \oplus \cdots \oplus (U_j \cap wV_{r_k})$$

and  $U_j \cap wV_{r_k} = \mathbb{F}e_{\beta+1} \oplus \cdots \oplus \mathbb{F}e_{\beta+k}$  for  $k = 1, \dots, \alpha_j$ . Write

$$ge_{\beta+k} = u_k + v_k$$

with  $u_k \in U_1 \oplus \cdots \oplus U_{j-1}$  and  $v_k \in U_j$ . Since  $g$  defines a linear isomorphism on the factor space  $(U_1 \oplus \cdots \oplus U_j)/(U_1 \oplus \cdots \oplus U_{j-1})$ , the map  $\ell : e_{\beta+k} \mapsto v_k$  defines a linear isomorphism on  $U_j$ . Hence  $\ell \in L_j$ . Since  $v_k \in \mathbb{F}e_{\beta+1} \oplus \cdots \oplus \mathbb{F}e_{\beta+k}$  for  $k = 1, \dots, \alpha_j$ , we have  $\ell \in L_j \cap B$ . Hence  $\ell^{-1}g \in wBw^{-1}$  by (i). Since  $\ell^{-1}ge_{\beta+k} = e_{\beta+k} + u_k$ , we have  $\ell^{-1}g \in N_j$ .  $\square$

Let  $H$  be a subgroup of  $L_j$  such that  $|H \backslash L_j / (L_j \cap B)| < \infty$  and  $Q = HN_j = N_jH$ .

**Proposition 6.5.** *Suppose  $L_j = \bigsqcup_{k=1}^{n_H} Hg_k(L_j \cap B)$ . Then*

$$PwB = \bigsqcup_{k=1}^{n_H} Qg_kwB.$$

*Proof.* By the map  $g \mapsto gw^{-1}$ ,

$$Q \backslash PwB / B \cong Q \backslash PwBw^{-1} / wBw^{-1}.$$

So we have only to consider the decomposition  $Q \backslash P / (P \cap wBw^{-1})$  of  $P$ . Since  $Q \supset N_i$ , we can take representatives in  $L_j$ . Suppose  $g_1, g_2 \in L_i$  satisfies  $g_1 \in Qg_2(P \cap wBw^{-1})$ . Then we have

$$g_1 = nhg_2\ell n'$$

with some  $n \in N_j, h \in H, \ell \in L_j \cap B$  and  $n' \in N_j \cap wBw^{-1}$  by Lemma 6.4. Since

$$hg_2\ell n' = n^{-1}g_1 = g_1(g_1^{-1}n^{-1}g_1) \in g_1N_j,$$

we have  $hg_2\ell = g_1$ . Hence  $Q \backslash P / (P \cap wBw^{-1}) \cong H \backslash L_j / (L_j \cap B)$ .  $\square$

Since  $|P \backslash G / B| = \frac{n!}{\alpha_1! \cdots \alpha_p!}$ , we have

**Corollary 6.6.**  $|Q \backslash G / B| = \frac{n_H n!}{\alpha_1! \cdots \alpha_p!}.$

The following result given by T. Hashimoto is useful.

**Proposition 6.7.** ([H04]) *Let  $H$  be a subgroup of  $\mathrm{GL}_n(\mathbb{F})$  of the form*

$$H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mid A \text{ is an upper triangular matrix in } \mathrm{GL}_{n-1}(\mathbb{F}) \right\}.$$

*Then there are a finite number of  $H$ -orbits on the full flag variety of  $\mathrm{GL}_n(\mathbb{F})$ .*

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FACULTY OF LETTERS, RYUKOKU UNIVERSITY, KYOTO 612-8577, JAPAN  
E-mail address: matsuki@let.ryukoku.ac.jp